

ON THE TENSOR PRODUCT OF LINEAR SITES AND GROTHENDIECK CATEGORIES

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ABSTRACT. We define a tensor product of linear sites, of which we investigate the functoriality properties. Consequently we define a tensor product of Grothendieck categories based upon their representations as categories of linear sheaves. We show that this tensor product is a special case of the tensor product of locally presentable linear categories, and we relate it to an α -version of the Deligne tensor product of abelian categories. We describe the tensor product of non-commutative projective schemes in terms of \mathbb{Z} -algebras, and show that for projective schemes our tensor product corresponds to the usual product scheme.

1. INTRODUCTION

A Grothendieck category \mathcal{C} is a cocomplete abelian category with a generator and exact filtered colimits. Grothendieck categories are arguably the most important large abelian categories, second only to module categories. They play an important role in non-commutative algebraic geometry, where they are used as models for non-commutative spaces since the work of Artin, Stafford, Van den Bergh and others ([3], [4], [38]). In algebraic geometry, one of the most basic operations to be performed with schemes X and Y is taking their product scheme $X \times Y$. For affine schemes $\text{Spec}(A)$ and $\text{Spec}(B)$, this corresponds to taking the tensor product $A \otimes B$ of the underlying rings. Our aim in this paper is to define a tensor product $\mathcal{C} \boxtimes \mathcal{D}$ for arbitrary Grothendieck categories \mathcal{C} and \mathcal{D} , such that for rings A and B we have

$$(1) \quad \text{Mod}(A) \boxtimes \text{Mod}(B) = \text{Mod}(A \otimes B).$$

As was originally shown in the Gabriel-Popescu theorem [33], Grothendieck categories are precisely the localizations of module categories. One way of seeing this, is by describing localizations of the category $\text{Mod}(A)$ of modules over a ring A by means of data on A , so called Gabriel topologies. In the Gabriel-Popescu theorem, the endomorphism ring of a generator of \mathcal{C} is endowed with such a Gabriel topology. Using the language of (linear) topologies on linear categories \mathfrak{a} , more generally one can characterize linear functors $\mathfrak{a} \rightarrow \mathcal{C}$ which induce an equivalence $\mathcal{C} \cong \text{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \subseteq \text{Mod}(\mathfrak{a})$, where $\mathcal{T}_{\mathfrak{a}}$ is a certain topology on \mathfrak{a} and $\text{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ is the category of linear sheaves on \mathfrak{a} with respect to this topology [26]. Our approach to the definition of a tensor product of Grothendieck categories consists of the following steps:

- (i) First, we define the tensor product of linear sites $(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ and $(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ to be $(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}})$ for a certain tensor product topology $\mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}$ on the standard tensor product of linear categories $\mathfrak{a} \otimes \mathfrak{b}$.

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(ii) Next, we show that the definition

$$(2) \quad \mathrm{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \boxtimes \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) = \mathrm{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}})$$

is a good definition for Grothendieck categories, as it is independent of the particular sites chosen in the sheaf category representations (up to equivalence of categories).

Step (i) is carried out in §2. The topologies $\mathcal{T}_{\mathfrak{a}}$ and $\mathcal{T}_{\mathfrak{b}}$ naturally give rise to two “one-sided” topologies \mathcal{T}_1 and \mathcal{T}_2 on $\mathfrak{a} \otimes \mathfrak{b}$, and we put $\mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}$ equal to the supremum of \mathcal{T}_1 and \mathcal{T}_2 in the lattice of topologies on $\mathfrak{a} \otimes \mathfrak{b}$ (Definition 2.12). We further describe the corresponding operations between localizing Serre subcategories, as well as between strict localizations. In particular, we show that

$$(3) \quad \mathrm{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}) = \mathrm{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_1) \cap \mathrm{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_2).$$

For compatible localizing Serre subcategories in the sense of [8], it is well known that their supremum is described by the Gabriel product, and using this description it is easily seen that the infimum of compatible strict localizations is simply their intersection. However, the general case is more subtle and our analysis is based upon the construction of a localizing hull (Proposition 2.2) for full subcategories closed under subquotients. This eventually leads to the proof of (3) in complete generality.

An application of our constructions to the strict localizations and localizing Serre subcategories corresponding to the linear sites associated to Quillen exact categories, recovers the constructions from [18], which inspired the current work (§2.8).

Step (ii) is based upon an analysis of the functoriality of our tensor product of sites, which is carried out in §3. An alternative approach making use of the already established tensor product of locally presentable categories going back to Kelly [21] [22] will be discussed in §5.4. Since the functoriality properties established in §3 are of independent interest in the context of non-commutative geometry, we present a complete proof of step (ii) without reference to local presentability, thus reflecting our own initial approach to the subject. A detailed discussion of the relation with the tensor product of locally presentable categories is contained in §5.1.

The classical notions of continuous and cocontinuous functors from [1] have their linear counterparts, and we show that these types of functors are preserved by the tensor product of sites. Our main interest goes out to a special type of functors $\phi : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \longrightarrow (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ between sites, which we call LC functors (the letters stand for “Lemme de comparaison”). Roughly speaking, ϕ satisfies (LC) (Definition 3.4) if:

- (1) ϕ is generating with respect to $\mathcal{T}_{\mathfrak{b}}$;
- (2) ϕ is fully faithful up to $\mathcal{T}_{\mathfrak{a}}$;
- (3) $\mathcal{T}_{\mathfrak{a}} = \phi^{-1}\mathcal{T}_{\mathfrak{b}}$.

The technical heart of the paper is the proof that our tensor product preserves LC functors (Proposition 3.14). Both the generating condition (1) and the fullness part of condition (2) are preserved separately. However, the faithfulness part is only preserved in combination with fullness (Lemma 3.12). This extends the situation for rings: surjections of rings are preserved under tensor product, injections are not (unless some flatness is assumed), but isomorphisms are obviously preserved by any functor hence also by tensoring.

The importance of LC functors $\phi : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \longrightarrow (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ lies in the fact that they induce equivalences of categories $\mathrm{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \cong \mathrm{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$. Further, any two representations of a given Grothendieck category \mathcal{C} as $\mathcal{C} \cong \mathrm{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ and $\mathcal{C} \cong \mathrm{Sh}(\mathfrak{a}', \mathcal{T}_{\mathfrak{a}'})$ can

be related through a roof of LC functors. This easily yields independence of (2) from the choice of sheaf category representations (Proposition 4.1).

In §4, we define the tensor product $\mathcal{C} \boxtimes \mathcal{D}$ for arbitrary Grothendieck categories \mathcal{C} and \mathcal{D} by formula (2) for arbitrary representations $\mathcal{C} \cong \mathrm{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ and $\mathcal{D} \cong \mathrm{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ (Definition 4.2). We apply the tensor product to \mathbb{Z} -algebras and schemes. In [5], [40], \mathbb{Z} -algebras are used as a tool to describe non-commutative deformations of projective planes and quadrics. They are closely related to the graded algebras turning up in projective geometry, but better suited for the purpose of algebraic deformation. In particular, under some finiteness conditions, they allow nice categories of “quasicoherent modules” [38], [32]. A (positively graded) \mathbb{Z} -algebra is a linear category \mathfrak{a} with $\mathrm{Ob}(\mathfrak{a}) = \mathbb{Z}$ and $\mathfrak{a}(n, m) = 0$ unless $n \geq m$. In [11], \mathbb{Z} -algebras \mathfrak{a} are endowed with a certain *tails topology* $\mathcal{T}_{\mathrm{tails}}$ and the category $\mathrm{Sh}(\mathfrak{a}, \mathcal{T}_{\mathrm{tails}})$ is proposed as a replacement for the category of quasicoherent modules, which exists in complete generality. We thus investigate the tensor product of two arbitrary tails sites $(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ and $(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ and show the existence of a cocontinuous functor

$$(4) \quad \Delta : ((\mathfrak{a} \otimes \mathfrak{b})_{\Delta}, \mathcal{T}_{\mathrm{tails}}) \longrightarrow (\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}})$$

from the natural diagonal \mathbb{Z} -algebra $(\mathfrak{a} \otimes \mathfrak{b})_{\Delta} \subseteq \mathfrak{a} \otimes \mathfrak{b}$ consisting of the objects (n, n) for $n \in \mathbb{Z}$ to the tensor site (Proposition 4.5). For a \mathbb{Z} -algebra \mathfrak{a} , the *degree* of an element in $\mathfrak{a}(n, m)$ is $n - m$ and we say that \mathfrak{a} is generated in degree 1 if every element can be written as a linear combination of products of elements of degree 1 (Definition 4.6). If \mathfrak{a} and \mathfrak{b} are generated in degree 1, then the functor Δ from (4) is actually an LC functor (Theorem 4.9). When applied to projective schemes X and Y , by looking at the \mathbb{Z} -algebras associated to defining graded algebras which are generated in degree 1, we obtain the following formula (Theorem 4.12):

$$(5) \quad \mathrm{Qch}(X) \boxtimes \mathrm{Qch}(Y) = \mathrm{Qch}(X \times Y).$$

Formula (5) is expected to hold in greater generality, at least for schemes and suitable stacks, which is work in progress.

In §5.1, we discuss the relation of our tensor product with other tensor products of categories in the literature. The existence of a tensor product of locally presentable categories goes back to [21], [22] and features in [2], [7], [9], [10]. It is well known that Grothendieck categories are locally presentable. For locally α -presentable Grothendieck categories, we use canonical sheaf representations in terms of the sites of α -presentable objects in order to calculate our tensor product, and we show that it coincides with the tensor product as locally presentable categories. In particular, the tensor product is again locally α -presentable. As a special case, we observe that locally finitely presentable Grothendieck categories are preserved under tensor product. In contrast, the stronger property of local coherence, which imposes the category of finitely presented objects to be abelian, is *not* preserved under tensor product, as is already seen for rings. Hence, one can view the tensor product of Grothendieck categories as a solution, within the framework of abelian categories, to the non-existence, in general, of the Deligne tensor product of small abelian categories. Indeed, it was shown by López Franco in [24] that the Deligne tensor product of abelian categories \mathcal{A} and \mathcal{B} from [13] exists precisely when the finitely cocomplete tensor product $\mathcal{A} \otimes_{\mathrm{fp}} \mathcal{B}$ is abelian, and this is the case precisely when the tensor product $\mathrm{Lex}(\mathcal{A}) \boxtimes \mathrm{Lex}(\mathcal{B})$ is locally coherent. As suggested to us by Henning Krause, we further examine the situation in terms of an α -Deligne tensor product of α -cocomplete abelian categories, showing that every tensor product of Grothendieck categories is accompanied by a parallel α -Deligne tensor product of its categories of α -presented objects for sufficiently large α .

Our tensor product can be seen as a k -linear counterpart to the *product* of Grothendieck toposes which is described by Johnstone in [16], [17] and its relation

with the tensor product of locally presentable categories is to some extent parallel to Pitts' work in [31]. We should note however that unlike in the case of toposes, working over $\text{Mod}(k)$ rather than over Set , the tensor product does not describe a 2-categorical product, but instead introduces a 2-categorical monoidal structure. Further, the functoriality properties we prove open up the possibility of describing a suitable monoidal 2-category of Grothendieck categories as a 2-localization of a monoidal 2-category of sites at the class of LC functors. This idea applies equally well to the Set -based setup. The details will appear in [35].

A combination of Pitts' approach and our description of the tensor product in terms of localizing Serre subcategories from §2.5 leads to a natural tensor product for well-generated algebraic triangulated categories, which stand in relation to derived categories of differential graded algebras like Grothendieck categories stand in relation to module categories [34]. The main idea is briefly sketched in §5.4, its development is work in progress [35].

In the future, it is our goal to apply tensor products of large categories (Grothendieck or enhanced triangulated) in order to widen the scope of the generalized Deligne conjecture proven in [36], [37].

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2. TENSOR PRODUCT OF LINEAR SITES

Throughout, let k be a commutative ground ring. For a k -linear category \mathfrak{a} , we put $\text{Mod}(\mathfrak{a}) = \text{Fun}_k(\mathfrak{a}^{\text{op}}, \text{Mod}(k))$, the category of k -linear functors from \mathfrak{a}^{op} to the category $\text{Mod}(k)$ of k -modules. Consider two k -linear categories \mathfrak{a} and \mathfrak{b} , with tensor product $\mathfrak{a} \otimes \mathfrak{b} = \mathfrak{a} \otimes_k \mathfrak{b}$. The starting point for our quest for a tensor product \boxtimes between Grothendieck abelian categories is the requirement that for module categories $\text{Mod}(\mathfrak{a})$ and $\text{Mod}(\mathfrak{b})$, we should have

$$(6) \quad \text{Mod}(\mathfrak{a}) \boxtimes \text{Mod}(\mathfrak{b}) = \text{Mod}(\mathfrak{a} \otimes \mathfrak{b}).$$

If we want to extend this principle to localizations of module categories, we should find a way of associating, to given localizations of $\text{Mod}(\mathfrak{a})$ and $\text{Mod}(\mathfrak{b})$, a new localization of $\text{Mod}(\mathfrak{a} \otimes \mathfrak{b})$. In this section, we detail three natural ways of doing this, based upon the following three isomorphic posets associated to the localization theory of $\text{Mod}(\mathfrak{c})$ for a linear category \mathfrak{c} (see §2.3):

- (1) The poset T of linear topologies on \mathfrak{c} ;
- (2) The poset W of localizing Serre subcategories of $\text{Mod}(\mathfrak{c})$;
- (3) The opposite poset L^{op} of the poset L of strict localizations of $\text{Mod}(\mathfrak{c})$.

More precisely, taking $\mathfrak{c} = \mathfrak{a} \otimes \mathfrak{b}$:

- (1) To topologies $\mathcal{T}_\mathfrak{a}$ on \mathfrak{a} and $\mathcal{T}_\mathfrak{b}$ on \mathfrak{b} , we associate “one-sided” topologies \mathcal{T}_1 (induced by $\mathcal{T}_\mathfrak{a}$) and \mathcal{T}_2 (induced by $\mathcal{T}_\mathfrak{b}$) on $\mathfrak{a} \otimes \mathfrak{b}$, and we put $\mathcal{T}_\mathfrak{a} \boxtimes \mathcal{T}_\mathfrak{b} = \mathcal{T}_1 \vee \mathcal{T}_2$ in T (see §2.4).
- (2) To localizing Serre subcategories $\mathcal{W}_\mathfrak{a} \subseteq \text{Mod}(\mathfrak{a})$ and $\mathcal{W}_\mathfrak{b} \subseteq \text{Mod}(\mathfrak{b})$, we associate the localizing Serre subcategories $\mathcal{W}_1, \mathcal{W}_2 \subseteq \text{Mod}(\mathfrak{a} \otimes \mathfrak{b})$ of objects which are in $\mathcal{W}_\mathfrak{a}$ (resp. $\mathcal{W}_\mathfrak{b}$) in the first (resp. second) variable, and we put $\mathcal{W}_\mathfrak{a} \boxtimes \mathcal{W}_\mathfrak{b} = \mathcal{W}_1 \vee \mathcal{W}_2$ in W (see §2.5). An explicit description is based upon the construction of the localizing hull for subcategories closed under subquotients from §2.2.

- (3) To strict localizations $\mathcal{L}_a \subseteq \text{Mod}(\mathfrak{a})$ and $\mathcal{L}_b \subseteq \text{Mod}(\mathfrak{b})$, we associate the strict localizations $\mathcal{L}_1, \mathcal{L}_2 \subseteq \text{Mod}(\mathfrak{a} \otimes \mathfrak{b})$ of objects which are in \mathcal{L}_a (resp. \mathcal{L}_b) in the first (resp. second) variable, and we put $\mathcal{L}_a \boxtimes \mathcal{L}_b = \mathcal{L}_1 \wedge \mathcal{L}_2$ in L (see §2.6). Using the relation between W and L , one sees that actually $\mathcal{L}_a \boxtimes \mathcal{L}_b = \mathcal{L}_1 \cap \mathcal{L}_2$.

From the order theoretic definitions of $\mathcal{T}_a \boxtimes \mathcal{T}_b$, $\mathcal{W}_a \boxtimes \mathcal{W}_b$ and $\mathcal{L}_a \boxtimes \mathcal{L}_b$, we conclude that in order to establish that they correspond under the isomorphisms between T , W and L^{op} , it suffices to establish the claim for \mathcal{T}_a , \mathcal{W}_a and \mathcal{L}_a (and similarly for \mathcal{T}_b , \mathcal{W}_b and \mathcal{L}_b). This is done in §2.7.

An application to Quillen exact categories recovers notions from [18] which inspired our definitions, as discussed in §2.8.

2.1. Linear sites. We will use the terminology and notations from [25, §2]. Let k be a commutative ground ring and let \mathfrak{a} be a small k -linear category. Every object $A \in \mathfrak{a}$ determines a representable \mathfrak{a} -module

$$\mathfrak{a}(-, A) : \mathfrak{a}^{\text{op}} \longrightarrow \text{Mod}(k) : B \longmapsto \mathfrak{a}(B, A).$$

A *sieve on A* is a submodule $R \subseteq \mathfrak{a}(-, A)$. A *cover system \mathcal{R}* on \mathfrak{a} consists of specifying, for every $A \in \mathfrak{a}$, a collection $\mathcal{R}(A)$ of sieves on A , called *covering sieves on A* or simply *covers of A* . One can list a number of properties a cover system can satisfy, as is done in [25, §2.2]. The most important properties are the *identity axiom*, the *pullback axiom*, and the *glueing axiom*. If \mathcal{R} satisfies the identity and pullback axioms, it is called a *localizing system*. If it moreover satisfies the glueing axiom, it is called a *topology*. Hence, what we call a topology is the k -linear counterpart of the notion of a Grothendieck topology.

Note that the intersection of a collection of topologies on \mathfrak{a} remains a topology, and \mathfrak{a} can be endowed with the *discrete* topology for which every sieve is covering. Hence, for an arbitrary cover system \mathcal{R} on \mathfrak{a} , there exists a smallest topology $\langle \mathcal{R} \rangle_{\text{top}}$ on \mathfrak{a} with $\mathcal{R} \subseteq \langle \mathcal{R} \rangle_{\text{top}}$. If \mathcal{R} is localizing, an explicit description of $\langle \mathcal{R} \rangle_{\text{top}}$ is available (see [25, §2.2]). Consequently, the poset T of topologies on \mathfrak{a} ordered by inclusion is a complete lattice with $\inf_i \mathcal{T}_i = \cap_i \mathcal{T}_i$ and $\sup_i \mathcal{T}_i = \langle \cup_i \mathcal{T}_i \rangle_{\text{top}}$.

2.2. Localizing subcategories. Let \mathcal{C} be a Grothendieck category. Recall that a *localizing Serre* subcategory (*localizing subcategory* for short) $\mathcal{W} \subseteq \mathcal{C}$ is a full subcategory closed under subquotients, extensions and coproducts. It follows in particular that a localizing subcategory \mathcal{W} is closed under colimits. As the intersection of localizing subcategories is again such, for every full subcategory $\mathcal{H} \subseteq \mathcal{C}$ there is a smallest localizing subcategory $\langle \mathcal{H} \rangle_{\text{loc}}$ with $\mathcal{H} \subseteq \langle \mathcal{H} \rangle_{\text{loc}}$, the *localizing hull* of \mathcal{H} . In particular, the poset W of localizing subcategories of \mathcal{C} is a complete lattice with $\inf_i \mathcal{W}_i = \cap_i \mathcal{W}_i$ and $\sup_i \mathcal{W}_i = \langle \cup_i \mathcal{W}_i \rangle_{\text{loc}}$. In this section we give an explicit description of $\langle \mathcal{H} \rangle_{\text{loc}}$ when \mathcal{H} is closed under subquotients.

Definition 2.1. Consider $\mathcal{H} \subseteq \text{Ob}(\mathcal{C})$ and $C \in \mathcal{C}$. An *ascending filtration* of C consists of an ordinal α and a collection of subobjects $(M_\beta)_{\beta \leq \alpha}$ of C such that $M_0 = 0$, $i \leq j$ implies $M_i \subseteq M_j$, $M_\beta = \cup_{\gamma < \beta} M_\gamma$ if β is a limit ordinal, and $M_\alpha = C$. An ascending filtration $(M_\beta)_{\beta \leq \alpha}$ of C is called an *\mathcal{H} -filtration* provided that $M_{\beta+1}/M_\beta \in \mathcal{H}$ for all $\beta < \alpha$, and in this case C is called *\mathcal{H} -filtered*.

Proposition 2.2. For $\mathcal{H} \subseteq \mathcal{C}$ closed under subquotients, $\langle \mathcal{H} \rangle_{\text{loc}}$ is the full subcategory of all \mathcal{H} -filtered objects.

Proof. Suppose first that $\mathcal{H} \subseteq \mathcal{W}$ for \mathcal{W} localizing. Consider an object $C \in \mathcal{C}$ with \mathcal{H} -filtration $(M_\beta)_{\beta \leq \alpha}$. We show by transfinite induction that every $M_\beta \in \mathcal{W}$. The statement is true for $M_0 = 0$. Suppose $M_\beta \in \mathcal{W}$. For $M_{\beta+1}$ we have an exact sequence $0 \longrightarrow M_\beta \longrightarrow M_{\beta+1} \longrightarrow M_{\beta+1}/M_\beta \longrightarrow 0$ so since \mathcal{W} is closed under

extensions we have $M_{\beta+1} \in \mathcal{W}$. For a limit ordinal β , we have $M_\beta \in \mathcal{W}$ since \mathcal{W} is closed under colimits.

Next we prove that the full subcategory of \mathcal{H} -filtered objects is closed under coproducts and extensions. Note that in this part of the proof we do not use that \mathcal{H} is closed under subquotients.

Consider a coproduct $C = \bigoplus_{i \in I} C_i$ of \mathcal{H} -filtered objects. We may safely assume that the coproduct is indexed by successor ordinals, that is $C = \bigoplus_{\gamma+1 < \alpha} C_{\gamma+1}$ for an ordinal α . We put $C_\alpha = 0$. We inductively define an ascending filtration $(D_\beta)_{\beta \leq \alpha}$ of C with $D_0 = 0$. For a successor ordinal $\gamma+1 \leq \alpha$, we put $D_{\gamma+1} = D_\gamma \oplus C_{\gamma+1}$ and for a limit ordinal $\beta \leq \alpha$ we put $D_\beta = \bigcup_{\gamma < \beta} D_\gamma$. Note that $D_\alpha = C$.

By assumption, every $C_{\beta+1}$ with $\beta+1 < \alpha$ has an \mathcal{H} -filtration $(M_\gamma^{\beta+1})_{\gamma \leq \alpha_{\beta+1}}$ for some ordinal $\alpha_{\beta+1}$. By transfinite induction on α we construct for every D_β with $\beta \leq \alpha$ an \mathcal{H} -filtration refining the chosen \mathcal{H} -filtrations of the D_γ with $\gamma < \beta$. We have the filtration $(D_0)_0$ for $D_0 = 0$. Suppose a \mathcal{H} -filtration $(P_\gamma^\beta)_{\gamma \leq \theta_\beta}$ is chosen for D_β with θ_β some ordinal. For a successor ordinal β , we have $D_{\beta+1} = D_\beta \oplus C_{\beta+1}$. We consider the ordinal sum $\theta_{\beta+1} = \theta_\beta + \alpha_{\beta+1}$. We join the two \mathcal{H} -filtrations together into an \mathcal{H} -filtration $(P_\gamma^{\beta+1})_{\gamma \leq \theta_{\beta+1}}$ with $P_\gamma^{\beta+1} = P_\gamma^\beta$ for $\gamma \leq \theta_\beta$ and $P_{\theta_\beta+\gamma}^{\beta+1} = D_\beta \oplus M_\gamma^{\beta+1}$ for $\gamma \leq \alpha_{\beta+1}$. For a limit ordinal $\beta \leq \alpha$, we put $\theta_\beta = \bigcup_{\gamma < \beta} \theta_\gamma$. We construct an \mathcal{H} -filtration $(P_\zeta^\beta)_{\zeta \leq \theta_\beta}$ of D_β . For $\zeta < \theta_\beta$, there exists $\gamma < \beta$ with $\zeta < \theta_\gamma$, and we put $P_\zeta^\beta = P_\zeta^\gamma$. This is well defined by construction. We further put $P_{\theta_\beta}^\beta = D_\beta$.

Next, consider an exact sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$

in \mathcal{C} and \mathcal{H} -filtrations $(M_\beta)_{\beta \leq \alpha'}$ of C' and $(N_\beta)_{\beta \leq \alpha''}$ of C'' . For the ordinal sum $\alpha = \alpha' + \alpha''$, we obtain an ascending filtration $(P_\gamma)_{\gamma \leq \alpha}$ of C with $P_\gamma = f(M_\gamma)$ for $\gamma \leq \alpha'$ and $P_{\alpha'+\gamma} = g^{-1}(N_\gamma)$ for $\gamma \leq \alpha''$. Note that we have $P_{\alpha'} = f(M_{\alpha'}) = g^{-1}(N_0)$ as desired since the sequence is exact. Further, we have $g^{-1}(N_{\gamma+1})/g^{-1}N_\gamma \cong N_{\gamma+1}/N_\gamma \in \mathcal{H}$.

To finish the proof, we show that the full subcategory of \mathcal{H} -filtered objects is closed under subquotients. Consider an exact sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$

in \mathcal{C} and an \mathcal{H} -filtration $(M_\beta)_{\beta \leq \alpha}$ of C . We obtain an ascending filtration $(f^{-1}(M_\beta))_{\beta \leq \alpha}$ of C' with canonical monomorphisms $f^{-1}(M_{\beta+1})/f^{-1}(M_\beta) \longrightarrow M_{\beta+1}/M_\beta$. Hence, since \mathcal{H} is closed under subobjects, this is an \mathcal{H} -filtration of C' . We obtain an ascending filtration $(g(M_\beta))_{\beta \leq \alpha}$ of C'' with canonical epimorphisms $M_{\beta+1}/M_\beta \longrightarrow g(M_{\beta+1})/g(M_\beta)$. Hence, since \mathcal{H} is closed under quotient objects, this is an \mathcal{H} -filtration of C'' . \square

Corollary 2.3. *In the lattice \mathcal{W} of localizing subcategories of \mathcal{C} , $\sup_i \mathcal{W}_i = \langle \bigcup_i \mathcal{W}_i \rangle_{\text{loc}}$ is given by the full subcategory of $\bigcup_i \mathcal{W}_i$ -filtered objects.*

Remark 2.4. Recall that two (localizing) Serre subcategories $\mathcal{W}_1, \mathcal{W}_2$ are *compatible* if $\mathcal{W}_1 * \mathcal{W}_2 = \mathcal{W}_2 * \mathcal{W}_1$ for the Gabriel product

$$\mathcal{W}_1 * \mathcal{W}_2 = \{C \in \mathcal{C} \mid \exists W_1 \in \mathcal{W}_1, W_2 \in \mathcal{W}_2, 0 \longrightarrow W_1 \longrightarrow C \longrightarrow W_2 \longrightarrow 0\}.$$

and in this case $\mathcal{W}_1 * \mathcal{W}_2$ is the smallest (localizing) Serre subcategory containing \mathcal{W}_1 and \mathcal{W}_2 . Note that in general, by Proposition 2.2 we have $\mathcal{W}_1 * \mathcal{W}_2 \subseteq \langle \mathcal{W}_1 \cup \mathcal{W}_2 \rangle_{\text{loc}}$, and we have equality if and only if \mathcal{W}_1 and \mathcal{W}_2 are compatible.

To end this section we describe the relation with orthogonal complements. We call an object C *left orthogonal* to an object D and D *right orthogonal* to C (notation $C \perp D$) provided that $\text{Ext}_{\mathcal{C}}^0(C, D) = 0 = \text{Ext}_{\mathcal{C}}^1(C, D)$. For a full subcategory $\mathcal{H} \subseteq \mathcal{C}$, we obtain the following full subcategories of \mathcal{C} :

- $\mathcal{H}^\perp = \{C \in \mathcal{C} \mid H \perp C \ \forall H \in \mathcal{H}\}$;
- ${}^\perp\mathcal{H} = \{C \in \mathcal{C} \mid C \perp H \ \forall H \in \mathcal{H}\}$

which are called the *right orthogonal complement* and the *left orthogonal complement* of \mathcal{H} respectively. Note that in the literature, different variations upon the terminology are used involving either Ext^0 or Ext^1 . The following statement for Ext^1 -orthogonals is known as Eklof's lemma, see for instance [30, Lemma 6.3.1] in the context of Grothendieck categories. The corresponding statement for Ext^0 -orthogonals is trivially verified. Hence, in our terminology, we have:

Proposition 2.5. *For a full subcategory $\mathcal{H} \subseteq \mathcal{C}$, the left orthogonal ${}^\perp\mathcal{H}$ is closed under ${}^\perp\mathcal{H}$ -filtered objects.*

Corollary 2.6. *Let $\mathcal{H} \subseteq \mathcal{C}$ be a full subcategory closed under subquotients. We have $(\langle \mathcal{H} \rangle_{\text{loc}})^\perp = \mathcal{H}^\perp$.*

Proof. This follows from Propositions 2.2 and 2.5. □

Corollary 2.7. *For localizing subcategories $(\mathcal{W}_i)_i$, we have $(\sup_i \mathcal{W}_i)^\perp = \cap_i \mathcal{W}_i^\perp$.*

Proof. This follows from Corollaries 2.3 and 2.6. □

2.3. Equivalent approaches to localization. Let \mathfrak{a} be a linear category and let \mathcal{C} be a Grothendieck category. Recall that a strict localization $\mathcal{L} \subseteq \mathcal{C}$ is a full subcategory which is closed under adding isomorphic objects, for which the inclusion functor $i : \mathcal{L} \rightarrow \mathcal{C}$ has an exact left adjoint $a : \mathcal{C} \rightarrow \mathcal{L}$. Consider the following posets, ordered by inclusion:

- (1) The poset T of linear topologies on \mathfrak{a} ;
- (2) The poset W of localizing Serre subcategories of \mathcal{C} ;
- (3) The poset L of strict localizations of \mathcal{C} .

It is well known that the data in (2) and (3) are equivalent, and for $\mathcal{C} = \text{Mod}(\mathfrak{a})$ all three types of data are equivalent. Let us briefly recall the isomorphisms involved. We have an order isomorphism between T and W , and dualities between T and L and between W and L respectively (the duality between W and L holds for arbitrary \mathcal{C}). We use the following notations. For $\mathcal{T} \in T$, $\mathcal{W}_{\mathcal{T}}$ and $\mathcal{L}_{\mathcal{T}}$ are the associated localizing subcategory and the associated localization. For $\mathcal{W} \in W$, $\mathcal{T}_{\mathcal{W}}$ and $\mathcal{L}_{\mathcal{T}}$ are the associated topology and the associated localization. For $\mathcal{L} \in L$, $\mathcal{T}_{\mathcal{L}}$ and $\mathcal{W}_{\mathcal{L}}$ are the associated topology and localizing subcategory.

We describe the involved constructions. Consider $\mathcal{T} \in T$. A module $F \in \text{Mod}(\mathfrak{a})$ is called a *sheaf* on \mathfrak{a} provided that $F(A) \cong \mathcal{C}(\mathfrak{a}(-, A), F) \rightarrow \mathcal{C}(R, F)$ is an isomorphism for all $R \in \mathcal{T}(A)$. We thus obtain the full subcategory $\text{Sh}(\mathfrak{a}, \mathcal{T})$ of sheaves on \mathfrak{a} and we have $\mathcal{L}_{\mathcal{T}} = \text{Sh}(\mathfrak{a}, \mathcal{T})$. A module $F \in \text{Mod}(\mathfrak{a})$ is called a *null presheaf* if for all $x \in F(A)$ there exists $R \in \mathcal{T}(A)$ such that for all $f : A' \rightarrow A$ in $R(A)$ we have $F(f)(x) = 0$. Then $\mathcal{W}_{\mathcal{T}}$ is the full subcategory of null presheaves.

Consider $\mathcal{W} \in W$. A subobject $R \subseteq \mathfrak{a}(-, A)$ is in $\mathcal{T}_{\mathcal{W}}(A)$ if and only if we have $\mathfrak{a}(-, A)/R \in \mathcal{W}$. We have $\mathcal{L}_{\mathcal{W}} = \mathcal{W}^\perp$.

Consider $\mathcal{L} \in L$ and let $a : \mathcal{C} \rightarrow \mathcal{L}$ be an exact left adjoint of the inclusion $\mathcal{L} \subseteq \mathcal{C}$. We have $\mathcal{W}_{\mathcal{L}} = \text{Ker}(a) = \{F \in \text{Mod}(\mathfrak{a}) \mid a(F) = 0\}$ and a sieve $r : R \rightarrow \mathfrak{a}(-, A)$ is in $\mathcal{T}_{\mathcal{L}}$ if and only if $a(r)$ is an isomorphism.

Let us first consider the duality between W and L in an arbitrary Grothendieck category \mathcal{C} . We obtain:

Proposition 2.8. *For a collection of strict localizations $(\mathcal{L}_i)_i$ in L , we have $\inf_i \mathcal{L}_i = \bigcap_i \mathcal{L}_i$.*

Proof. It suffices that $\bigcap_i \mathcal{L}_i$ is a strict localization, which follows from Corollary 2.7 after writing $\mathcal{L}_i = \mathcal{W}_i^\perp$ for the corresponding localizing subcategories \mathcal{W}_i . \square

Next consider the order isomorphism between T and W for $\mathcal{C} = \text{Mod}(\mathfrak{a})$. Since it respects suprema, we have:

Proposition 2.9. $\mathcal{W}_{\sup_i \mathcal{T}_i} = \sup_i \mathcal{W}_{\mathcal{T}_i}$ and $\mathcal{T}_{\sup_i \mathcal{W}_i} = \sup_i \mathcal{T}_{\mathcal{W}_i}$.

2.4. The tensor product topology. Consider linear sites $(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$ and $(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$. In this section we define a topology $\mathcal{T} = \mathcal{T}_\mathfrak{a} \boxtimes \mathcal{T}_\mathfrak{b}$ on $\mathfrak{c} = \mathfrak{a} \otimes \mathfrak{b}$, called the tensor product topology. For $M \in \text{Mod}(\mathfrak{a})$ and $N \in \text{Mod}(\mathfrak{b})$, we obtain $M \otimes N \in \text{Mod}(\mathfrak{a} \otimes \mathfrak{b})$ with $(M \otimes N)(A, B) = M(A) \otimes N(B)$. Consider objects $A \in \mathfrak{a}$, $B \in \mathfrak{b}$ and covering sieves $R \in \mathcal{T}_\mathfrak{a}(A)$ and $S \in \mathcal{T}_\mathfrak{b}(B)$. We have $\mathfrak{a}(-, A) \otimes \mathfrak{b}(-, B) = \mathfrak{c}(-, (A, B))$ and we thus obtain a canonical morphism

$$\phi_{R,S} : R \otimes S \longrightarrow \mathfrak{c}(-, (A, B)).$$

We define the *tensor product sieve* of R and S to be

$$R \boxtimes S = \text{Im}(\phi_{R,S}).$$

Concretely, any element in $(R \boxtimes S)(A', B')$ can be written as $\sum_{i=1}^n \alpha_i \otimes \beta_i$ with $\alpha_i \in R(A')$ and $\beta_i \in S(B')$. Consider the following cover systems on $\mathfrak{a} \otimes \mathfrak{b}$:

- $\mathcal{R}_\mathfrak{a} = \{R \boxtimes \mathfrak{b}(-, B) \mid R \in \mathcal{T}_\mathfrak{a}, B \in \mathfrak{b}\}$;
- $\mathcal{R}_\mathfrak{b} = \{\mathfrak{a}(-, A) \boxtimes S \mid S \in \mathcal{T}_\mathfrak{b}, A \in \mathfrak{a}\}$;
- $\mathcal{R} = \{R \boxtimes S \mid R \in \mathcal{T}_\mathfrak{a}, S \in \mathcal{T}_\mathfrak{b}\}$.

Lemma 2.10. *Consider objects $A, A' \in \mathfrak{a}$ and $B, B' \in \mathfrak{b}$, covering sieves $R \in \mathcal{T}_\mathfrak{a}(A)$ and $S \in \mathcal{T}_\mathfrak{b}(B)$, and a morphism $h = \sum_{i=1}^n f_i \otimes g_i \in \mathfrak{a}(A', A) \otimes \mathfrak{b}(B', B) = \mathfrak{c}((A', B'), (A, B))$. We have*

$$(\bigcap_{i=1}^n f_i^{-1} R) \boxtimes (\bigcap_{i=1}^n g_i^{-1} S) \subseteq h^{-1}(R \boxtimes S).$$

Lemma 2.11. *Consider objects $A \in \mathfrak{a}$ and $B \in \mathfrak{b}$, a covering sieve $R \in \mathcal{T}_\mathfrak{a}(A)$, and for every morphism $h = \sum_{i=1}^n f_i \otimes g_i \in (R \boxtimes \mathfrak{b}(-, B))(A', B')$ with $A' \in \mathfrak{a}$ and $B' \in \mathfrak{b}$ a covering sieve $R_h \in \mathcal{T}_\mathfrak{a}(A')$. We have*

$$(R \circ (R_{f \otimes 1})_f) \boxtimes \mathfrak{b}(-, B) \subseteq (R \boxtimes \mathfrak{b}(-, B)) \circ (R_h \boxtimes \mathfrak{b}(-, B'))_h.$$

Definition 2.12. The *tensor product topology* $\mathcal{T} = \mathcal{T}_\mathfrak{a} \boxtimes \mathcal{T}_\mathfrak{b}$ on $\mathfrak{a} \otimes \mathfrak{b}$ is the smallest topology containing $\mathcal{R}_\mathfrak{a}$ and $\mathcal{R}_\mathfrak{b}$, that is

$$\mathcal{T}_\mathfrak{a} \boxtimes \mathcal{T}_\mathfrak{b} = \langle \mathcal{R}_\mathfrak{a} \cup \mathcal{R}_\mathfrak{b} \rangle_{\text{top}}.$$

The *tensor product site* of $(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$ and $(\mathfrak{b}, \mathcal{T}_\mathfrak{b})$ is

$$(\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \boxtimes (\mathfrak{b}, \mathcal{T}_\mathfrak{b}) = (\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_\mathfrak{a} \boxtimes \mathcal{T}_\mathfrak{b}) = (\mathfrak{c}, \mathcal{T}).$$

Proposition 2.13.

- (1) *The cover systems $\mathcal{T}_1 = \mathcal{R}_\mathfrak{a}^{\text{up}}$ and $\mathcal{T}_2 = \mathcal{R}_\mathfrak{b}^{\text{up}}$ are topologies.*
- (2) *The cover systems \mathcal{R}^{up} and $\mathcal{T}_1 \cup \mathcal{T}_2$ are upclosed and localizing.*
- (3) *The topology \mathcal{T} is the smallest topology containing \mathcal{R} .*
- (4) *We have $\mathcal{T} = \mathcal{R}^{\text{upglue}}$ and $\mathcal{T} = (\mathcal{R}_\mathfrak{a} \cup \mathcal{R}_\mathfrak{b})^{\text{upglue}}$.*

Proof. The cover systems \mathcal{T}_1 , \mathcal{T}_2 , $\mathcal{T}_1 \cup \mathcal{T}_2$ and \mathcal{R}^{up} are localizing by Lemma 2.10 and \mathcal{T}_1 and \mathcal{T}_2 are topologies by further invoking Lemma 2.11. It remains to prove (3). Obviously $\mathcal{R}_\mathfrak{a} \cup \mathcal{R}_\mathfrak{b} \subseteq \mathcal{R}$ so it remains to show that $\mathcal{R} \subseteq \mathcal{T}$. For $R \boxtimes S \in \mathcal{R}(A, B)$, we consider $R \boxtimes \mathfrak{b}(-, B) \in \mathcal{R}_\mathfrak{a}(A, B)$. For every $h = \sum_{i=1}^n f_i \otimes g_i \in (R \boxtimes \mathfrak{b}(-, B))(A', B')$, we have $\mathfrak{a}(-, A') \boxtimes (\bigcap_{i=1}^n g_i^{-1} S) \subseteq h^{-1}(R \boxtimes S)$ so $R \boxtimes S \in \mathcal{T}(A, B)$ by the glueing property. \square

In the lattice T of topologies on $\mathfrak{a} \otimes \mathfrak{b}$, we have

$$(7) \quad \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}} = \mathcal{T}_1 \vee \mathcal{T}_2.$$

2.5. Tensor product of localizing subcategories. Consider linear categories \mathfrak{a} and \mathfrak{b} with $\mathfrak{c} = \mathfrak{a} \otimes \mathfrak{b}$ and full subcategories $\mathcal{W}_{\mathfrak{a}} \subseteq \text{Mod}(\mathfrak{a})$ and $\mathcal{W}_{\mathfrak{b}} \subseteq \text{Mod}(\mathfrak{b})$.

Consider the following full subcategories of $\text{Mod}(\mathfrak{c})$:

- $\mathcal{W}_1 = \{F \in \text{Mod}(\mathfrak{c}) \mid F(-, B) \in \mathcal{W}_{\mathfrak{a}} \ \forall B \in \mathfrak{b}\}$;
- $\mathcal{W}_2 = \{F \in \text{Mod}(\mathfrak{c}) \mid F(A, -) \in \mathcal{W}_{\mathfrak{b}} \ \forall A \in \mathfrak{a}\}$.

Proposition 2.14. *If $\mathcal{W}_{\mathfrak{a}}$ (resp. $\mathcal{W}_{\mathfrak{b}}$) is closed under extensions, coproducts, sub-objects or quotient objects, then so is \mathcal{W}_1 (resp. \mathcal{W}_2).*

We define the *tensor product* of localizing subcategories $\mathcal{W}_{\mathfrak{a}}$ and $\mathcal{W}_{\mathfrak{b}}$ to be

$$\mathcal{W}_{\mathfrak{a}} \boxtimes \mathcal{W}_{\mathfrak{b}} = \langle \mathcal{W}_1 \cup \mathcal{W}_2 \rangle_{\text{loc}}.$$

In the lattice W of localizing subcategories in $\text{Mod}(\mathfrak{c})$, we thus have

$$(8) \quad \mathcal{W}_{\mathfrak{a}} \boxtimes \mathcal{W}_{\mathfrak{b}} = \mathcal{W}_1 \vee \mathcal{W}_2.$$

2.6. Tensor product of strict localizations. Let \mathfrak{a} , \mathfrak{b} and $\mathfrak{c} = \mathfrak{a} \otimes \mathfrak{b}$ be as before. Consider strict localizations $i_{\mathfrak{a}} : \mathcal{L}_{\mathfrak{a}} \rightarrow \text{Mod}(\mathfrak{a})$ and $i_{\mathfrak{b}} : \mathcal{L}_{\mathfrak{b}} \rightarrow \text{Mod}(\mathfrak{b})$ with exact left adjoints $a_{\mathfrak{a}} : \text{Mod}(\mathfrak{a}) \rightarrow \mathcal{L}_{\mathfrak{a}}$ and $a_{\mathfrak{b}} : \text{Mod}(\mathfrak{b}) \rightarrow \mathcal{L}_{\mathfrak{b}}$. Consider the following full subcategories of $\text{Mod}(\mathfrak{c})$:

- $\mathcal{L}_1 = \{F \in \text{Mod}(\mathfrak{c}) \mid F(-, B) \in \mathcal{L}_{\mathfrak{a}} \ \forall B \in \mathfrak{b}\}$;
- $\mathcal{L}_2 = \{F \in \text{Mod}(\mathfrak{c}) \mid F(A, -) \in \mathcal{L}_{\mathfrak{b}} \ \forall A \in \mathfrak{a}\}$.

The natural functors

- $a_1 : \text{Mod}(\mathfrak{c}) \rightarrow \mathcal{L}_1 : F \mapsto (a_1(F) : (A, B) \mapsto (a_{\mathfrak{a}}(F(-, B)))(A))$;
- $a_2 : \text{Mod}(\mathfrak{c}) \rightarrow \mathcal{L}_2 : F \mapsto (a_2(F) : (A, B) \mapsto (a_{\mathfrak{b}}(F(A, -)))(B))$

are readily seen to be exact left adjoints of the inclusions $i_1 : \mathcal{L}_1 \rightarrow \text{Mod}(\mathfrak{c})$ and $i_2 : \mathcal{L}_2 \rightarrow \text{Mod}(\mathfrak{c})$ respectively.

We define the *tensor product localization* of $\mathcal{L}_{\mathfrak{a}}$ and $\mathcal{L}_{\mathfrak{b}}$ to be

$$\mathcal{L}_{\mathfrak{a}} \boxtimes \mathcal{L}_{\mathfrak{b}} = \mathcal{L}_1 \cap \mathcal{L}_2,$$

which is a strict localization by Proposition 2.8. In the lattice L of strict localizations of $\text{Mod}(\mathfrak{c})$, we thus have

$$(9) \quad \mathcal{L}_{\mathfrak{a}} \boxtimes \mathcal{L}_{\mathfrak{b}} = \mathcal{L}_1 \wedge \mathcal{L}_2.$$

2.7. Relation between the three tensor products. Let \mathfrak{a} , \mathfrak{b} and $\mathfrak{c} = \mathfrak{a} \otimes \mathfrak{b}$ be as before. Suppose \mathfrak{a} is endowed with a topology $\mathcal{T}_{\mathfrak{a}}$, a localizing subcategory $\mathcal{W}_{\mathfrak{a}}$ and a strict localization $\mathcal{L}_{\mathfrak{a}}$ (with left adjoint $a_{\mathfrak{a}} : \text{Mod}(\mathfrak{a}) \rightarrow \mathcal{L}$) which correspond as in §2.3, an similarly \mathfrak{b} is endowed with a topology $\mathcal{T}_{\mathfrak{b}}$, a localizing subcategory $\mathcal{W}_{\mathfrak{b}}$ and a strict localization $\mathcal{L}_{\mathfrak{b}}$ (with left adjoint $a_{\mathfrak{b}} : \text{Mod}(\mathfrak{b}) \rightarrow \mathcal{L}$). Our aim in this section is to show that $\mathcal{T} = \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}$, $\mathcal{W} = \mathcal{W}_{\mathfrak{a}} \boxtimes \mathcal{W}_{\mathfrak{b}}$ and $\mathcal{L} = \mathcal{L}_{\mathfrak{a}} \boxtimes \mathcal{L}_{\mathfrak{b}}$ (with left adjoint $a : \text{Mod}(\mathfrak{c}) \rightarrow \mathcal{L}$) correspond as well.

Let us first look at the relation between \mathcal{W} and \mathcal{L} . We first note the following:

Proposition 2.15. *The localizing subcategory \mathcal{W}_1 (resp. \mathcal{W}_2) and the strict localization \mathcal{L}_1 (resp. \mathcal{L}_2) correspond under the isomorphism between W and L .*

Proof. Consider $\mathcal{W}_{\mathfrak{a}} = \text{Ker}(a_{\mathfrak{a}})$ and $\mathcal{W}_{\mathfrak{b}} = \text{Ker}(a_{\mathfrak{b}})$ and let $\mathcal{W}_1, \mathcal{W}_2$ be as defined in §2.5. By direct inspection, we have $\text{Ker}(a_1) = \mathcal{W}_1$ and $\text{Ker}(a_2) = \mathcal{W}_2$. \square

Corollary 2.16. *The localizing subcategory $\mathcal{W}_{\mathfrak{a}} \boxtimes \mathcal{W}_{\mathfrak{b}}$ and the strict localization $\mathcal{L}_{\mathfrak{a}} \boxtimes \mathcal{L}_{\mathfrak{b}}$ correspond under the isomorphism between W and L .*

Proof. This follows from Proposition 2.15 and equations (8) and (9). \square

Next we look at the relation between \mathcal{T} and \mathcal{W} . Again, we must first establish the relation between \mathcal{T}_1 and \mathcal{W}_1 and between \mathcal{T}_2 and \mathcal{W}_2 .

Proposition 2.17. *The topology \mathcal{T}_1 and the localizing subcategory \mathcal{W}_1 (resp. the topology \mathcal{T}_2 and the localizing subcategory \mathcal{W}_2) correspond under the isomorphism between T and L .*

Proof. It suffices to show the following inclusions: (1) $\mathcal{T}_1 \subseteq \mathcal{T}_{\mathcal{W}_1}$, (2) $\mathcal{W}_1 \subseteq \mathcal{W}_{\mathcal{T}_1}$.

For (1) consider, for $R \in \mathcal{T}_a(A)$ and $B \in \mathfrak{b}$, the sieve $R \boxtimes \mathfrak{b}(-, B)$. The exact sequence $R \rightarrow \mathfrak{a}(-, A) \rightarrow \mathfrak{a}(-, A)/R \rightarrow 0$ gives rise to the exact sequence $R \otimes \mathfrak{b}(-, B) \rightarrow \mathfrak{c}(-, (A, B)) \rightarrow (\mathfrak{a}(-, A)/R) \otimes \mathfrak{b}(-, B) \rightarrow 0$ and hence $F = \mathfrak{c}(-, (A, B))/(R \boxtimes \mathfrak{b}(-, B)) \cong (\mathfrak{a}(-, A)/R) \otimes \mathfrak{b}(-, B)$. It suffices to show that $F \in \mathcal{W}_1$. Now we have $Z = \mathfrak{a}(-, A)/R \in \mathcal{W}_a$ and for $B' \in \mathfrak{b}$, $F(-, B') = Z \otimes \mathfrak{b}(B', B)$ is a colimit of objects in \mathcal{W}_a , hence it is itself in \mathcal{W}_a as desired.

For (2), consider $F \in \mathcal{W}_1$ and $x \in F(A, B) = F(-, B)(A)$. Since $F(-, B) \in \mathcal{W}_a$, there exists $R \in \mathcal{T}_a$ such that for all $f \in R(A')$ we have $F(f \otimes 1) : F(A, B) \rightarrow F(A', B) : x \mapsto 0$. It now suffices to consider $R \boxtimes \mathfrak{b}(-, B) \in \mathcal{T}_1(A, B)$. For every element $h = \sum_i f_i \otimes g_i = \sum_i (1 \otimes g_i)(f_i \otimes 1) \in (R \boxtimes \mathfrak{b}(-, B))(A', B')$, we have $F(h)(x) = 0$ as desired. \square

Corollary 2.18. *The topology $\mathcal{T}_a \boxtimes \mathcal{T}_b$ and the localizing subcategory $\mathcal{W}_a \boxtimes \mathcal{W}_b$ correspond under the isomorphism between T and W .*

Proof. This follows from Proposition 2.17 and equations (7) and (8). \square

2.8. Exact categories. The following setup from [18] inspired our definitions. Let \mathfrak{a} be an exact category in the sense of Quillen, and let $\text{Lex}(\mathfrak{a}) \subseteq \text{Mod}(\mathfrak{a})$ be the full subcategory of left exact functors, that is additive functors $F : \mathfrak{a}^{\text{op}} \rightarrow \text{Mod}(k)$ which send conflations $0 \rightarrow K \rightarrow D \rightarrow C \rightarrow 0$ to short exact sequences $0 \rightarrow F(C) \rightarrow F(D) \rightarrow F(K)$. The category \mathfrak{a} can be endowed with the single deflation topology, for which a sieve is covering if and only if it contains a deflation for the exact structure. The category of sheaves for this topology is precisely $\text{Lex}(\mathfrak{a})$, and the corresponding localizing Serre subcategory \mathcal{W}_a is the category of weakly effaceable modules, that is, modules $M \in \text{Mod}(\mathfrak{a})$ such that for every $x \in M(A)$ there exists a deflation $A' \rightarrow A$ for which $M(A) \rightarrow M(A')$ maps x to zero.

Now let \mathfrak{a} and \mathfrak{b} be exact categories. In [18, §2.6], the full category $\text{Lex}(\mathfrak{b}, \mathfrak{a}^{\text{op}}) \subseteq \text{Mod}(\mathfrak{b} \otimes \mathfrak{a}^{\text{op}})$ of bimodules M for which every $M(-, A) \in \text{Lex}(\mathfrak{b})$ and every $M(B, -) \in \text{Lex}(\mathfrak{a}^{\text{op}})$ for the natural “dual” exact structure on \mathfrak{a}^{op} is introduced. With the definition from §2.6, we thus have

$$\text{Lex}(\mathfrak{b}, \mathfrak{a}^{\text{op}}) = \text{Lex}(\mathfrak{b}) \boxtimes \text{Lex}(\mathfrak{a}^{\text{op}})$$

and in [18, Prop. 2.22], the relation with the localizing Serre subcategory $\mathcal{W}_b \boxtimes \mathcal{W}_a^{\text{op}}$ from Corollary 2.16 was demonstrated using the description of the Gabriel product.

In [18], it is argued that $\text{Lex}(\mathfrak{b}, \mathfrak{a}^{\text{op}})$ is the correct bimodule category to consider between exact categories, where we look at bimodules contravariant in $B \in \mathfrak{b}$ and covariant in $A \in \mathfrak{a}$. In particular, it is shown that over a field k , Hochschild cohomology of \mathfrak{a} in the sense of [20], and of $\text{Lex}(\mathfrak{a})$ in the sense of [27] is equal to

$$HH^n(\mathfrak{a}) = \text{Ext}_{\text{Lex}(\mathfrak{a}, \mathfrak{a}^{\text{op}})}^n(1_{\mathfrak{a}}, 1_{\mathfrak{a}}).$$

It is not clear how this approach could be extended to more general sites, as it makes essential use of the existence of a natural “dual site” for the site associated to an exact category.

3. FUNCTORIALITY OF THE TENSOR PRODUCT OF LINEAR SITES

Let \mathfrak{a} and \mathfrak{b} be k -linear categories as before. Let us return to the starting point for our quest for a tensor product \boxtimes between Grothendieck abelian categories, namely the requirement that

$$(10) \quad \text{Mod}(\mathfrak{a}) \boxtimes \text{Mod}(\mathfrak{b}) = \text{Mod}(\mathfrak{a} \otimes \mathfrak{b}).$$

Using the 2-categorical structure of the category $\text{Cat}(k)$ of k -linear categories, functors and natural transformations, it is not hard to see that \boxtimes can be defined based upon (10) for module categories \mathcal{C} . A module category \mathcal{C} is intrinsically characterized by the existence of a set of finitely generated projective generators, and different choices of generators give rise to Morita equivalent linear categories. For Morita bimodules M between \mathfrak{a} and \mathfrak{a}' and N between \mathfrak{b} and \mathfrak{b}' , it is readily seen that $M \otimes N$ with $M \otimes N((A, B), (A', B')) = M(A, A') \otimes N(B, B')$ defines a Morita bimodule between $\mathfrak{a} \otimes \mathfrak{b}$ and $\mathfrak{a}' \otimes \mathfrak{b}'$. Our aim in this section is to develop the necessary tools in order to extend this situation from module categories to arbitrary Grothendieck categories. Rather than focussing on bimodules, we first focus on functors between sites. The underlying idea is that any equivalence between sheaf categories can be represented by a roof of LC functors between sites, where an LC functor is a particular kind of functor which induces an equivalence between sheaf categories. Roughly speaking, an LC functor $\phi : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightarrow (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$ is generating with respect to $\mathcal{T}_{\mathfrak{c}}$, fully faithful up to $\mathcal{T}_{\mathfrak{a}}$, and has $\phi^{-1}\mathcal{T}_{\mathfrak{c}} = \mathcal{T}_{\mathfrak{a}}$ (Definition 3.4). The main result of this section is that LC functors are preserved under tensor product of sites (Proposition 3.14).

3.1. Functors. Consider linear categories \mathfrak{a} and \mathfrak{c} and a linear functor $\phi : \mathfrak{a} \rightarrow \mathfrak{c}$. Suppose $\mathcal{T}_{\mathfrak{a}}$ and $\mathcal{T}_{\mathfrak{c}}$ are cover systems on the respective categories.

Definition 3.1. Suppose $\mathcal{T}_{\mathfrak{a}}$ and $\mathcal{T}_{\mathfrak{c}}$ are localizing. The functor $\phi : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightarrow (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$ is called *continuous* provided that $\phi^* : \text{Mod}(\mathfrak{c}) \rightarrow \text{Mod}(\mathfrak{a})$ preserves sheaves, and hence restricts to a functor $\phi_s : \text{Sh}(\mathfrak{c}, \mathcal{T}_{\mathfrak{c}}) \rightarrow \text{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$.

Definition 3.2. [25, Def. 2.11] The functor $\phi : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightarrow (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$ is called *cocontinuous* provided that for every $A \in \mathfrak{a}$ and $R \in \mathcal{T}_{\mathfrak{c}}(\phi(A))$ there exists $S \in \mathcal{T}_{\mathfrak{a}}(A)$ with $\phi S \subseteq R$.

Remark 3.3. Suppose $\mathcal{T}_{\mathfrak{a}}$ and $\mathcal{T}_{\mathfrak{c}}$ are topologies. Continuous morphisms are the linear counterpart of the continuous morphisms from [1], and cocontinuous morphisms are the linear counterpart of the cocontinuous morphisms from [1]. In [25], the term “cover continuous” is used for what we call cocontinuous here.

Next we recall some special conditions (see [25, §2.5]).

Definition 3.4. Consider a linear functor $\phi : \mathfrak{a} \rightarrow \mathfrak{c}$.

- (1) Suppose \mathfrak{c} is endowed with a cover system $\mathcal{T}_{\mathfrak{c}}$. We say that $\phi : \mathfrak{a} \rightarrow (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$ satisfies
 - (G) if for every $C \in \mathfrak{c}$ there is a covering family $(\phi(A_i) \rightarrow C)_i$ for $\mathcal{T}_{\mathfrak{c}}$.
- (2) Suppose \mathfrak{a} is endowed with a cover system $\mathcal{T}_{\mathfrak{a}}$. We say that $\phi : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightarrow \mathfrak{c}$ satisfies
 - (F) if for every $c : \phi(A) \rightarrow \phi(A')$ in \mathfrak{c} there exists a covering family $a_i : A_i \rightarrow A$ for $\mathcal{T}_{\mathfrak{a}}$ and $f_i : A_i \rightarrow A'$ with $c\phi(a_i) = \phi(f_i)$;
 - (FF) if for every $a : A \rightarrow A'$ in \mathfrak{a} with $\phi(a) = 0$ there exists a covering family $a_i : A_i \rightarrow A$ for $\mathcal{T}_{\mathfrak{a}}$ with $aa_i = 0$.
- (3) Suppose \mathfrak{a} and \mathfrak{c} are endowed with cover systems $\mathcal{T}_{\mathfrak{a}}$ and $\mathcal{T}_{\mathfrak{c}}$ respectively. We say that $\phi : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightarrow (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$ satisfies
 - (LC) if ϕ satisfies (G) with respect to $\mathcal{T}_{\mathfrak{c}}$, (F) and (FF) with respect to $\mathcal{T}_{\mathfrak{a}}$, and we further have $\mathcal{T}_{\mathfrak{a}} = \phi^{-1}\mathcal{T}_{\mathfrak{c}}$.

We have the following ‘‘Lemme de comparaison’’ (see [1], [26], [25]):

Theorem 3.5. *If $\phi : (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \longrightarrow (\mathfrak{c}, \mathcal{T}_\mathfrak{c})$ satisfies (LC) and $\mathcal{T}_\mathfrak{c}$ is a topology, then $\mathcal{T}_\mathfrak{a}$ is a topology, ϕ is continuous and $\phi_s : \mathbf{Sh}(\mathfrak{c}, \mathcal{T}_\mathfrak{c}) \longrightarrow \mathbf{Sh}(\mathfrak{a}, \mathcal{T}_\mathfrak{a})$ is an equivalence of categories.*

The following lemma, which is easily proven by induction, will be used later on:

Lemma 3.6. *Suppose the functor $\phi : (\mathfrak{a}, \mathcal{T}_\mathfrak{a}) \longrightarrow (\mathfrak{c}, \mathcal{T}_\mathfrak{c})$ satisfies (F) and $\mathcal{T}_\mathfrak{a}$ is a topology and consider morphisms $c_i : \phi(A) \longrightarrow \phi(A')$ for $i = 1, \dots, n$. There exists a collection of morphisms $h_j : A_i \longrightarrow A$ for $j \in J$ with $\langle h_j \rangle \in \mathcal{T}_\mathfrak{a}(A)$ and $g_{ij} : A_i \longrightarrow A'$ such that $c_i \phi(h_j) = \phi(g_{ij})$ for all $i = 1, \dots, n$ and $j \in J$.*

3.2. Tensor product of functors. Consider linear categories \mathfrak{a} , \mathfrak{b} , \mathfrak{c} and \mathfrak{d} and linear functors $\phi : \mathfrak{a} \longrightarrow \mathfrak{c}$ and $\psi : \mathfrak{b} \longrightarrow \mathfrak{d}$. Consider the tensor product functor

$$\phi \otimes \psi : \mathfrak{a} \otimes \mathfrak{b} \longrightarrow \mathfrak{c} \otimes \mathfrak{d}.$$

We have $\phi \otimes \psi = (1 \otimes \psi)(\phi \otimes 1)$ for

$$\phi \otimes 1 : \mathfrak{a} \otimes \mathfrak{b} \longrightarrow \mathfrak{c} \otimes \mathfrak{b}$$

and

$$1 \otimes \psi : \mathfrak{c} \otimes \mathfrak{b} \longrightarrow \mathfrak{c} \otimes \mathfrak{d}.$$

Suppose $\mathcal{T}_\mathfrak{a}$, $\mathcal{T}_\mathfrak{b}$, $\mathcal{T}_\mathfrak{c}$ and $\mathcal{T}_\mathfrak{d}$ are cover systems on the respective categories.

Proposition 3.7. *Suppose all cover systems are localizing. If ϕ and ψ are continuous, then so is $\phi \otimes \psi$.*

Proof. We have to look at the functor $(\phi \otimes \psi)^* : \mathbf{Mod}(\mathfrak{c} \otimes \mathfrak{d}) \longrightarrow \mathbf{Mod}(\mathfrak{a} \otimes \mathfrak{b})$. According to §2.6, §2.7 we have $F \in \mathbf{Sh}(\mathfrak{c} \otimes \mathfrak{d}, \mathcal{T}_\mathfrak{c} \boxtimes \mathcal{T}_\mathfrak{d})$ if and only if $F(-, -)$ is a sheaf in both variables for $\mathcal{T}_\mathfrak{c}$ and $\mathcal{T}_\mathfrak{d}$ respectively. It readily follows that $(\phi \otimes \psi)^* F = F(\phi(-), \psi(-))$ is a sheaf in the first variable for $\mathcal{T}_\mathfrak{a}$ as soon as ϕ is continuous, and a sheaf in the second variable for $\mathcal{T}_\mathfrak{b}$ as soon as ψ is continuous. \square

The following is easy to check:

Lemma 3.8. *Consider $A \in \mathfrak{a}$, $B \in \mathfrak{b}$, and sieves $R \subseteq \mathfrak{a}(-, A)$ and $S \subseteq \mathfrak{b}(-, B)$. As sieves on $(\phi(A), B)$, we have*

$$\phi(R) \boxtimes S = (\phi \otimes 1)(R \boxtimes S).$$

Proposition 3.9. *If ϕ and ψ are cocontinuous, then so is $\phi \otimes \psi$.*

Proof. Since cocontinuous functors are stable under composition, it suffices to consider $\psi = 1 : (\mathfrak{b}, \mathcal{T}_\mathfrak{b}) \longrightarrow (\mathfrak{b}, \mathcal{T}_\mathfrak{b})$. We are to show that $\phi \otimes 1 : (\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_\mathfrak{a} \boxtimes \mathcal{T}_\mathfrak{b}) \longrightarrow (\mathfrak{c} \otimes \mathfrak{b}, \mathcal{T}_\mathfrak{c} \boxtimes \mathcal{T}_\mathfrak{b})$ is cocontinuous. By [25, Lem. 2.12], it suffices to show that $\phi \otimes 1$ is cocontinuous with respect to the localizing cover systems $\mathcal{L}_{\mathfrak{a}, \mathfrak{b}} = \{R \boxtimes S \mid R \in \mathcal{T}_\mathfrak{a}, S \in \mathcal{T}_\mathfrak{b}\}^{\text{up}}$ and $\mathcal{L}_{\mathfrak{c}, \mathfrak{b}} = \{T \boxtimes S \mid T \in \mathcal{T}_\mathfrak{c}, S \in \mathcal{T}_\mathfrak{b}\}^{\text{up}}$. Thus, consider $T \boxtimes S$ with $T \in \mathcal{T}_\mathfrak{c}(\phi(A))$, $S \in \mathcal{T}_\mathfrak{b}(B)$. By the assumption there exists $R \in \mathcal{T}_\mathfrak{a}$ with $\phi R \subseteq T$. Consequently, by Lemma 3.8 we have $(\phi \otimes 1)(R \boxtimes S) \subseteq T \boxtimes S$ as desired. \square

Lemma 3.10. *Suppose the functor ϕ satisfies (G) with respect to $\mathcal{T}_\mathfrak{c}$. Then the functor $\phi \otimes 1$ satisfies (G) with respect to $\mathcal{T}_\mathfrak{c} \boxtimes \mathcal{T}_\mathfrak{b}$.*

Proof. Consider $(C, B) \in \mathfrak{c} \otimes \mathfrak{b}$. There exists $R = \langle f_i : \phi(A_i) \longrightarrow C \rangle \in \mathcal{T}_\mathfrak{c}(C)$. It is easily seen that $R \boxtimes \mathfrak{b}(-, B) = \langle f_i \otimes 1_B : (\phi \otimes 1)(A_i, B) \longrightarrow (C, B) \rangle$. \square

Lemma 3.11. *Suppose the functor ϕ satisfies (F) with respect to $\mathcal{T}_\mathfrak{a}$. Then the functor $\phi \otimes 1$ satisfies (F) with respect to $\mathcal{T}_\mathfrak{a} \boxtimes \mathcal{T}_\mathfrak{b}$.*

Proof. Consider a morphism $h = \sum_{i=1}^n c_i \otimes b_i : (\phi(A), B) \longrightarrow (\phi(A'), B')$. We proceed by induction. Suppose we have a collection $(a_\alpha : A_\alpha \longrightarrow A)_\alpha$ of morphisms in \mathfrak{a} with $\langle a_\alpha \rangle \in \mathcal{T}_\mathfrak{a}(A)$ (and hence $\langle a_\alpha \otimes 1_B \rangle \in \mathcal{T}_\mathfrak{a} \boxtimes \mathcal{T}_\mathfrak{b}(A, B)$) and for $i \in \{1, \dots, m-1\}$ with $m \leq n$ and α , there exists $g_\alpha^i : (A_\alpha, B) \longrightarrow (A', B)$ with $(\phi \otimes 1)(g_\alpha^i) = (c_i \otimes b_i)(\phi(a_\alpha) \otimes 1_B)$. We show that the same holds for $i = m$. To this end, we consider $(c_m \otimes b_m)(\phi(a_\alpha) \otimes 1_B) = c_m \phi(a_\alpha) \otimes b_m$. For $c_m \phi(a_\alpha) : \phi(A_\alpha) \longrightarrow \phi(A')$, since ϕ satisfies (F) there is a collection $(a_\beta^\alpha : A_\beta^\alpha \longrightarrow A_\alpha)_\beta$ with $\langle a_\beta^\alpha \rangle \in \mathcal{T}_\mathfrak{a}(A_\alpha)$ and there exist morphisms $f_\beta^\alpha : A_\beta^\alpha \longrightarrow A'$ with $\phi(f_\beta^\alpha) = c_m \phi(a_\alpha) \phi(a_\beta^\alpha)$. Consequently, the collection of compositions $a_\alpha a_\beta^\alpha : A_\beta^\alpha \longrightarrow A$ are such that $\langle a_\alpha a_\beta^\alpha \rangle \in \mathcal{T}_\mathfrak{a}(A)$ by the glueing property (and hence $\langle a_\alpha a_\beta^\alpha \otimes 1_B \rangle \in \mathcal{T}_\mathfrak{a} \boxtimes \mathcal{T}_\mathfrak{b}(A, B)$). For $i \in \{1, \dots, m-1\}$, we have $(\phi \otimes 1)(g_\alpha^i(a_\beta^\alpha \otimes 1_B)) = (c_i \otimes b_i)(\phi(a_\alpha a_\beta^\alpha) \otimes 1_B)$ by the induction hypothesis. For m , we have $(\phi \otimes 1)(f_\beta^\alpha \otimes b_m) = (c_m \otimes b_m)(\phi(a_\alpha a_\beta^\alpha) \otimes 1_B)$ as desired. \square

Lemma 3.12. *Suppose the functor ϕ satisfies (F) and (FF) with respect to $\mathcal{T}_\mathfrak{a}$. Then the functor $\phi \otimes 1$ satisfies (F) and (FF) with respect to $\mathcal{T}_\mathfrak{a} \boxtimes \mathcal{T}_\mathfrak{b}$.*

Proof. Consider a morphism $h = \sum_{i=1}^n a_i \otimes b_i : (A, B) \longrightarrow (A', B')$ such that $0 = (\phi \otimes 1)(h) = \sum_{i=1}^n \phi(a_i) \otimes b_i$. Let $(c_\lambda)_{\lambda \in \Lambda}$ be a collection of generators of the k -module $\mathfrak{c}(\phi(A), \phi(A'))$ such that $\{1, \dots, n\} \subseteq \Lambda$ and $c_i = \phi(a_i)$ for $i \in \{1, \dots, n\}$. Put $b_\lambda = 0$ for $\lambda \in \Lambda \setminus \{1, \dots, n\}$. We thus have $0 = \sum_{\lambda \in \Lambda} c_\lambda \otimes b_\lambda \in \mathfrak{c}(\phi(A), \phi(A')) \otimes \mathfrak{b}(B, B')$. According to [14, Lem. 6.4], there exist $\bar{b}_j \in \mathfrak{b}(B, B')$ for $j \in J$ and $\kappa_{\lambda,j} \in k$ such that $(\kappa_{\lambda,j})_{\lambda \in \Lambda, j \in J}$ contains only finitely many non-zero elements, such that the following hold:

- (1) $b_\lambda = \sum_j \kappa_{\lambda,j} \bar{b}_j$ for all $\lambda \in \Lambda$;
- (2) $0 = \sum_\lambda \kappa_{\lambda,j} c_\lambda = \sum_{i=1}^n \kappa_{i,j} \phi(a_i) + \sum_{\lambda \in \Lambda \setminus \{1, \dots, n\}} \kappa_{\lambda,j} c_\lambda$ for all $j \in J$.

Using (F) for ϕ , we will first realize the right hand side of (2) as being in the image of ϕ up to a covering. Let $\Lambda_0 \subseteq \Lambda$ contain those λ 's for which there exists $j \in J$ with $\kappa_{\lambda,j} \neq 0$. Hence Λ_0 is finite. By Lemma 3.6, there exists a collection $h_\sigma : A_\sigma \longrightarrow A$ for $\sigma \in \Sigma$ with $\langle h_\sigma \rangle \in \mathcal{T}_\mathfrak{a}(A)$ and $g_{\lambda,\sigma} : A_\sigma \longrightarrow A'$ such that $c_\lambda \phi(h_\sigma) = \phi(g_{\lambda,\sigma})$ for $\lambda \in \Lambda_0$ and $\sigma \in \Sigma$. Further, we may clearly suppose that

$$(11) \quad g_{i,\sigma} = a_i h_\sigma$$

for $i = 1, \dots, n$. Hence, for $j \in J$ and $\sigma \in \Sigma$, from (2) we obtain:

$$0 = \sum_\lambda \kappa_{\lambda,j} c_\lambda \phi(h_\sigma) = \phi\left(\sum_\lambda \kappa_{\lambda,j} g_{\lambda,\sigma}\right).$$

Using (FF) for ϕ , for every $\sigma \in \Sigma$ we obtain a collection $h_\omega^\sigma : A_\omega^\sigma \longrightarrow A_\sigma$ for $\omega \in \Omega_\sigma$ with $\langle h_\omega^\sigma \rangle \in \mathcal{T}_\mathfrak{a}(A_\sigma)$ such that for every $\omega \in \Omega_\sigma$

$$(12) \quad 0 = \sum_\lambda \kappa_{\lambda,j} g_{\lambda,\sigma} h_\omega^\sigma.$$

Now consider the collection $h_\sigma h_\omega^\sigma : A_\omega^\sigma \longrightarrow A$ for $\sigma \in \Sigma$ and $\omega \in \Omega_\sigma$. By the glueing property we have $\langle h_\sigma h_\omega^\sigma \rangle \in \mathcal{T}_\mathfrak{a}(A)$. Further, we have $\langle h_\sigma h_\omega^\sigma \otimes 1_B \rangle \in \mathcal{T}_\mathfrak{a} \boxtimes \mathcal{T}_\mathfrak{b}(A, B)$. We claim that h becomes zero on this covering sieve of (A, B) . We have $h = \sum_{i=1}^n a_i \otimes b_i = \sum_j \left(\sum_{i=1}^n \kappa_{i,j} a_i\right) \otimes \bar{b}_j$. We compute

$$h(h_\sigma h_\omega^\sigma \otimes 1) = \sum_{j \in J} \left(\sum_{i=1}^n \kappa_{i,j} a_i h_\sigma h_\omega^\sigma \otimes \bar{b}_j\right).$$

Consider the expressions

$$x = \sum_{j \in J} \left(\sum_{\lambda \in \Lambda} \kappa_{\lambda,j} g_{\lambda,\sigma} h_\omega^\sigma \otimes \bar{b}_j\right)$$

and

$$y = \sum_{j \in J} \left(\sum_{\lambda \in \Lambda \setminus \{1, \dots, n\}} \kappa_{\lambda, j} g_{\lambda, \sigma} h_{\omega}^{\sigma} \otimes \bar{b}_j \right) = \sum_{\lambda \in \Lambda \setminus \{1, \dots, n\}} (g_{\lambda, \sigma} h_{\omega}^{\sigma} \otimes \sum_{j \in J} \kappa_{\lambda, j} \bar{b}_j).$$

Using equation (11), we clearly have

$$x = h(h_{\sigma} h_{\omega}^{\sigma} \otimes 1) + y.$$

By equation (12), we have $x = 0$. By definition and by condition (2) above, for $\lambda \in \Lambda \setminus \{1, \dots, n\}$, we have $0 = b_{\lambda} = \sum_{j \in J} \kappa_{\lambda, j} \bar{b}_j$ so also $y = 0$. We conclude that $h(h_{\sigma} h_{\omega}^{\sigma} \otimes 1) = 0$ as desired. \square

Lemma 3.13. *If the functor $\phi : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightarrow (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$ satisfies (LC), then so does the functor $\phi \otimes 1 : (\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}) \rightarrow (\mathfrak{c} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{c}} \boxtimes \mathcal{T}_{\mathfrak{b}})$.*

Proof. By Lemmas 3.10, 3.11, and 3.12, $\phi \otimes 1$ satisfies (G), (F) and (FF). We have $\mathcal{T}_{\mathfrak{a}} = \phi^{-1} \mathcal{T}_{\mathfrak{c}}$, and it remains to show that

$$\mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}} = (\phi \otimes 1)^{-1}(\mathcal{T}_{\mathfrak{c}} \boxtimes \mathcal{T}_{\mathfrak{b}}).$$

To prove the inclusion $\mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}} \subseteq (\phi \otimes 1)^{-1}(\mathcal{T}_{\mathfrak{c}} \boxtimes \mathcal{T}_{\mathfrak{b}})$, it suffices to look at a sieve $R \boxtimes S$ with $\phi(R) \in \mathcal{T}_{\mathfrak{c}}$ and $S \in \mathcal{T}_{\mathfrak{b}}$. It immediately follows from Lemma 3.8 that $(\phi \otimes 1)(R \boxtimes S) \in \mathcal{T}_{\mathfrak{c}} \boxtimes \mathcal{T}_{\mathfrak{b}}$.

For the other inclusion, by [25, Prop. 2.16], it suffices to show that $\phi \otimes 1 : (\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}) \rightarrow (\mathfrak{c} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{c}} \boxtimes \mathcal{T}_{\mathfrak{b}})$ is cocontinuous. By [25, Lem. 2.15], ϕ is cocontinuous, whence it follows by Proposition 3.9 that $\phi \otimes 1$ is cocontinuous as desired. \square

Proposition 3.14. *If the functors $\phi : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightarrow (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$ and $\psi : (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \rightarrow (\mathfrak{d}, \mathcal{T}_{\mathfrak{d}})$ both satisfy (G) (resp. (F), resp. (F) and (FF), resp. (LC)), then so does the functor $\phi \otimes \psi : (\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}) \rightarrow (\mathfrak{c} \otimes \mathfrak{d}, \mathcal{T}_{\mathfrak{c}} \boxtimes \mathcal{T}_{\mathfrak{d}})$.*

4. TENSOR PRODUCT OF GROTHENDIECK CATEGORIES

Based upon the previous sections, in §4.1 we are finally in a position to define the tensor product of Grothendieck categories $\mathcal{C} \cong \text{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ and $\mathcal{D} \cong \text{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}})$ to be given by

$$\mathcal{C} \boxtimes \mathcal{D} = \text{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}).$$

Functoriality of the tensor product of sites ensures that $\mathcal{C} \boxtimes \mathcal{D}$ is well-defined up to equivalence of categories. For an alternative approach ensuring well-definedness by making use of the already established tensor product of locally presentable categories, we refer to §5.4.

The remainder of this section is devoted to an application of our tensor product to \mathbb{Z} -algebras and schemes. In §4.2 we provide a nice realisation of the tensor product of \mathbb{Z} -algebras, while in §4.3, we show that for projective schemes X and Y we have

$$(13) \quad \text{Qch}(X) \boxtimes \text{Qch}(Y) = \text{Qch}(X \times Y).$$

This result generalizes to non-commutative projective schemes, and our proof is actually based upon the results in §4.2. Here, we use \mathbb{Z} -algebras as models for non-commutative schemes following [5], [40], [38], [11], and to a \mathbb{Z} -algebra \mathfrak{a} we can associate a certain category $\text{Qch}(\mathfrak{a})$ which replaces the quasicohherent sheaves, and which is obtained as a linear sheaf category with respect to a certain topology. For two \mathbb{Z} -algebras \mathfrak{a} and \mathfrak{b} generated in degree 1, there is a naturally associated diagonal \mathbb{Z} -algebra $(\mathfrak{a} \otimes \mathfrak{b})_{\Delta}$, for which we show that

$$(14) \quad \text{Qch}(\mathfrak{a}) \boxtimes \text{Qch}(\mathfrak{b}) \cong \text{Qch}((\mathfrak{a} \otimes \mathfrak{b})_{\Delta}).$$

The relation between (13) and (14) is provided by graded algebras (generated in degree 1), which on the one hand are used to represent schemes through the Proj construction, and which on the other hand give rise to associated \mathbb{Z} -algebras.

4.1. Tensor product of Grothendieck categories. Let \mathcal{C} be a k -linear Grothendieck category and let $(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ be a k -linear site. We say that a k -linear functor $u : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightarrow \mathcal{C}$ satisfies (LC), or is an *LC morphism* provided that $u : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightarrow (\mathcal{C}, \mathcal{T}_{\mathcal{C}})$ satisfies (LC) where $\mathcal{T}_{\mathcal{C}}$ is the topology of jointly epimorphic sieves. Precisely, $R \in \mathcal{T}_{\mathcal{C}}(\mathcal{C})$ if and only if $\bigoplus_{(f: C_f \rightarrow \mathcal{C}) \in R} C_f \rightarrow \mathcal{C}$ is an epimorphism. The general Gabriel-Popescu theorem states that for $\mathcal{T}_{\mathfrak{a}} = u^{-1}\mathcal{T}_{\mathcal{C}}$, we have that u is an LC morphism if and only if $\mathcal{T}_{\mathfrak{a}}$ is a topology and u gives rise to an equivalence $\mathcal{C} \rightarrow \text{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}})$ (see [26]).

Consider k -linear Grothendieck categories \mathcal{C} and \mathcal{D} .

Proposition 4.1. *Consider LC morphisms $u : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightarrow \mathcal{C}$, $u' : (\mathfrak{a}', \mathcal{T}_{\mathfrak{a}'}) \rightarrow \mathcal{C}$, $v : (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \rightarrow \mathcal{D}$, $v' : (\mathfrak{b}', \mathcal{T}_{\mathfrak{b}'}) \rightarrow \mathcal{D}$. There exists an equivalence of categories*

$$\text{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}) \cong \text{Sh}(\mathfrak{a}' \otimes \mathfrak{b}', \mathcal{T}_{\mathfrak{a}'} \boxtimes \mathcal{T}_{\mathfrak{b}'}).$$

Proof. Let $\mathfrak{c} \subseteq \mathcal{C}$ be the full subcategory with $\text{Ob}(\mathfrak{c}) = \{u(A) \mid A \in \mathfrak{a}\} \cup \{u'(A) \mid A' \in \mathfrak{a}'\}$ and let $\mathfrak{d} \subseteq \mathcal{D}$ be the full subcategory with $\text{Ob}(\mathfrak{d}) = \{v(B) \mid B \in \mathfrak{b}\} \cup \{v'(B) \mid B' \in \mathfrak{b}'\}$. Put $\mathcal{T}_{\mathfrak{c}} = i^{-1}\mathcal{T}_{\mathcal{C}}$ for the inclusion $i : \mathfrak{c} \rightarrow \mathcal{C}$ and the canonical topology $\mathcal{T}_{\mathcal{C}}$ on \mathcal{C} and put $\mathcal{T}_{\mathfrak{d}} = j^{-1}\mathcal{T}_{\mathcal{D}}$ for the inclusion $j : \mathfrak{d} \rightarrow \mathcal{D}$ and the canonical topology $\mathcal{T}_{\mathcal{D}}$ on \mathcal{D} . It follows that the induced functors $\bar{u} : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightarrow (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$, $\bar{u}' : (\mathfrak{a}', \mathcal{T}_{\mathfrak{a}'}) \rightarrow (\mathfrak{c}, \mathcal{T}_{\mathfrak{c}})$, $\bar{v} : (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \rightarrow (\mathfrak{d}, \mathcal{T}_{\mathfrak{d}})$, $\bar{v}' : (\mathfrak{b}', \mathcal{T}_{\mathfrak{b}'}) \rightarrow (\mathfrak{d}, \mathcal{T}_{\mathfrak{d}})$ are all LC morphisms. By Proposition 3.14, it follows that $\bar{u} \otimes \bar{v} : (\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}) \rightarrow (\mathfrak{c} \otimes \mathfrak{d}, \mathcal{T}_{\mathfrak{c}} \boxtimes \mathcal{T}_{\mathfrak{d}})$ and $\bar{u}' \otimes \bar{v}' : (\mathfrak{a}' \otimes \mathfrak{b}', \mathcal{T}_{\mathfrak{a}'} \boxtimes \mathcal{T}_{\mathfrak{b}'}) \rightarrow (\mathfrak{c} \otimes \mathfrak{d}, \mathcal{T}_{\mathfrak{c}} \boxtimes \mathcal{T}_{\mathfrak{d}})$ are LC morphisms, and in particular we have equivalences of categories $\text{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}) \cong \text{Sh}(\mathfrak{c} \otimes \mathfrak{d}, \mathcal{T}_{\mathfrak{c}} \boxtimes \mathcal{T}_{\mathfrak{d}}) \cong \text{Sh}(\mathfrak{a}' \otimes \mathfrak{b}', \mathcal{T}_{\mathfrak{a}'} \boxtimes \mathcal{T}_{\mathfrak{b}'})$. \square

Thanks to Proposition 4.1, we can now make the following definition:

Definition 4.2. Consider Grothendieck categories \mathcal{C} and \mathcal{D} . The *tensor product* $\mathcal{C} \boxtimes \mathcal{D}$ is the following Grothendieck category, defined up to equivalence of categories: for arbitrary LC morphisms $u : (\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \rightarrow \mathcal{C}$ and $v : (\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) \rightarrow \mathcal{D}$, we put

$$\mathcal{C} \boxtimes \mathcal{D} = \text{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}).$$

4.2. Tensor product of \mathbb{Z} -algebras. Recall that a \mathbb{Z} -algebra is a linear category \mathfrak{a} with $\text{Ob}(\mathfrak{a}) = \mathbb{Z}$. We further suppose that \mathfrak{a} is *positively graded*, that is $\mathfrak{a}(n, m) = 0$ for $n < m$. Following [11], we consider the sieves $\mathfrak{a}(-, m)_{\geq n} \subseteq \mathfrak{a}(-, m)$ for $n \geq m \in \mathbb{Z}$ with

$$\mathfrak{a}(l, m)_{\geq n} = \begin{cases} \mathfrak{a}(l, m) & \text{if } l \geq n \\ 0 & \text{otherwise} \end{cases}$$

and we consider the *tails localizing system*

$$\mathcal{L}_{\text{tails}} = \{\mathfrak{a}(-, m)_{\geq n} \mid n \geq m\}^{\text{up}}$$

and the *tails topology*

$$\mathcal{T}_{\text{tails}} = \mathcal{L}_{\text{tails}}^{\text{upglue}}.$$

Remark 4.3. In many cases of interest, we have $\mathcal{L}_{\text{tails}} = \mathcal{T}_{\text{tails}}$. This is the case for a noetherian \mathbb{Z} -algebra or for a connected, finitely generated \mathbb{Z} -algebra in the sense of [11].

For \mathbb{Z} -algebras \mathfrak{a} and \mathfrak{b} , we define the *diagonal* \mathbb{Z} -algebra $\mathfrak{c} = (\mathfrak{a} \otimes \mathfrak{b})_{\Delta}$ with

$$\mathfrak{c}(n, m) = (\mathfrak{a} \otimes \mathfrak{b})((n, n), (m, m)) = \mathfrak{a}(n, m) \otimes \mathfrak{b}(n, m).$$

There is a corresponding fully faithful functor

$$\Delta : \mathfrak{c} \longrightarrow \mathfrak{a} \otimes \mathfrak{b} : n \longmapsto (n, n).$$

Let $\mathcal{L}_a, \mathcal{L}_b, \mathcal{L}_c$ denote the tails localizing systems on $\mathfrak{a}, \mathfrak{b}$ and \mathfrak{c} respectively, and let $\mathcal{T}_a, \mathcal{T}_b, \mathcal{T}_c$ denote the corresponding tails topologies. Further, consider the following cover system on $\mathfrak{a} \otimes \mathfrak{b}$:

$$\mathcal{L}_{\mathfrak{a} \otimes \mathfrak{b}} = \{R \boxtimes S \mid R \in \mathcal{L}_a, S \in \mathcal{L}_b\}^{\text{up}}$$

Lemma 4.4. *The cover system $\mathcal{L}_{\mathfrak{a} \otimes \mathfrak{b}}$ is localizing and upclosed and $\mathcal{T}_a \boxtimes \mathcal{T}_b = \mathcal{L}_{\mathfrak{a} \otimes \mathfrak{b}}^{\text{upglue}}$.*

Proposition 4.5. *The functor $\Delta : (\mathfrak{c}, \mathcal{T}_c) \longrightarrow (\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_a \boxtimes \mathcal{T}_b)$ is cocontinuous and we have*

$$\Delta^{-1}(\mathcal{T}_a \boxtimes \mathcal{T}_b) \subseteq \mathcal{T}_c.$$

Proof. The second claim follows from the first one by [25, Prop. 2.16]. According to [25, Lem. 2.12], it suffices to prove the statement for \mathcal{L}_c on \mathfrak{c} and $\mathcal{L}_{\mathfrak{a} \otimes \mathfrak{b}}$ on $\mathfrak{a} \otimes \mathfrak{b}$. Hence, consider $m \in \mathfrak{c}$ and $\Delta(m) = (m, m) \in \mathfrak{a} \otimes \mathfrak{b}$, and $R = \mathfrak{a}(-, m)_{\geq n_1} \boxtimes \mathfrak{b}(-, m)_{\geq n_2} \in \mathcal{L}_{\mathfrak{a} \otimes \mathfrak{b}}(m, m)$. For $n = \max(n_1, n_2)$, consider $S = \mathfrak{c}(-, m)_{\geq n} \in \mathcal{L}_c$. We have $S(l) = \mathfrak{a}(l, m)_{\geq n} \otimes \mathfrak{b}(l, m)_{\geq n} \subseteq \mathfrak{a}(l, m)_{\geq n_1} \otimes \mathfrak{b}(l, m)_{\geq n_2} = R(l, l)$ so $\Delta S \subseteq R$ as desired. \square

In order to improve upon Proposition 4.5, we look at generation of \mathbb{Z} -algebras in the sense of [11].

- Definition 4.6.**
- (1) A linear category \mathfrak{a} is *generated* by subsets $X(A, A') \subseteq \mathfrak{a}(A, A')$ if every element of \mathfrak{a} can be written as a linear sum of products of elements in X .
 - (2) A \mathbb{Z} algebra \mathfrak{a} is *generated in certain degrees* $D \subseteq \mathbb{N}$ if it is generated by X with $X(n, m) = \emptyset$ unless $n - m \in D$.
 - (3) A \mathbb{Z} -algebra \mathfrak{a} is *finitely generated* if it is generated by X such that for all m the set $\cup_{d \in \mathbb{N}} X(m + d, m)$ is finite.
 - (4) A \mathbb{Z} -algebra \mathfrak{a} is *connected* if $\mathfrak{a}(n, n) \cong k$ for all n .

We make the following observation:

Proposition 4.7. *Consider \mathbb{Z} -algebras \mathfrak{a} and \mathfrak{b} and put $\mathfrak{c} = (\mathfrak{a} \otimes \mathfrak{b})_{\Delta}$.*

- (1) *If \mathfrak{a} is generated by X_a and \mathfrak{b} is generated by X_b , then \mathfrak{c} is generated by X_c with $X_c(n, m) = X_a(n, m) \times X_b(n, m)$.*
- (2) *If \mathfrak{a} and \mathfrak{b} are generated in degrees D (resp. finitely generated, resp. connected), then so is \mathfrak{c} .*

Remark 4.8. It was shown in [11] that for a connected, finitely generated \mathbb{Z} -algebra \mathfrak{a} , we have $\mathcal{L}_{\text{tails}} = \mathcal{T}_{\text{tails}}$.

Our main result is the following:

Theorem 4.9. *Consider \mathbb{Z} -algebras \mathfrak{a} and \mathfrak{b} which are generated in degree 1. The functor $\Delta : (\mathfrak{c}, \mathcal{T}_c) \longrightarrow (\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_a \boxtimes \mathcal{T}_b)$ satisfies (LC). In particular, we have $\mathcal{T}_c = \Delta^{-1}(\mathcal{T}_a \boxtimes \mathcal{T}_b)$ and*

$$\text{Sh}(\mathfrak{a}, \mathcal{T}_a) \boxtimes \text{Sh}(\mathfrak{b}, \mathcal{T}_b) = \text{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_a \boxtimes \mathcal{T}_b) \cong \text{Sh}(\mathfrak{c}, \mathcal{T}_c).$$

Proof. This follows from Proposition 4.5 and Lemmas 4.10 and 4.11. \square

Lemma 4.10. *Suppose the \mathbb{Z} -algebras \mathfrak{a} and \mathfrak{b} are generated in degree 1. The functor $\Delta : \mathfrak{c} \longrightarrow (\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_a \boxtimes \mathcal{T}_b)$ satisfies (G).*

Proof. Consider $(m_1, m_2) \in \mathfrak{a} \otimes \mathfrak{b}$. Suppose for instance that $m_2 \geq m_1$. Consider the cover $\mathfrak{a}(-, m_1)_{\geq m_2} \boxtimes \mathfrak{b}(-, m_2) \in \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}(m_1, m_2)$. We claim that this cover is generated by the morphisms $x \otimes 1 \in \mathfrak{a}(m_2, m_1) \otimes \mathfrak{b}(m_2, m_2)$ from the diagonal element (m_2, m_2) to (m_1, m_2) . Indeed, for an element $a \otimes b \in \mathfrak{a}(l_1, m_1) \otimes \mathfrak{b}(l_2, m_2)$ with $l_1 \geq m_2$, by the hypothesis on \mathfrak{a} we can write $a = \sum_{i=1}^k a'_i a''_i$ for $a'_i \in \mathfrak{a}(m_2, m_1)$ and $a''_i \in \mathfrak{a}(l_1, m_2)$. Hence, $a \otimes b = \sum_{i=1}^k (a'_i \otimes 1)(a''_i \otimes b)$ as desired. \square

Lemma 4.11. *Suppose the \mathbb{Z} -algebras \mathfrak{a} and \mathfrak{b} which are generated in degree 1. We have $\mathcal{T}_{\mathfrak{c}} \subseteq \Delta^{-1}(\mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}})$.*

Proof. Note that since Δ satisfies (G), (F) and (FF) with respect to $\mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}$ and $\Delta^{-1}(\mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}})$, it follows by [25, Thm. 2.13] that the cover system $\Delta^{-1}(\mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}})$ is a topology. Hence, to prove the desired inclusion, it suffices to show that $\mathcal{L}_{\mathfrak{c}} \subseteq \Delta^{-1}(\mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}})$. Consider $S = \mathfrak{c}(-, m)_{\geq n} \in \mathcal{L}_{\mathfrak{c}}(m)$. We are to show that $\Delta S \in \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}}(m, m)$. Now ΔS is generated by the morphisms in $\mathfrak{a}(l, m) \otimes \mathfrak{b}(l, m)$ for $l \geq n$. We claim that $\Delta S = \mathfrak{a}(-, m)_{\geq n} \boxtimes \mathfrak{b}(-, m)_{\geq n}$. To this end, we take an element $a \otimes b \in \mathfrak{a}(l_1, m) \otimes \mathfrak{b}(l_2, m)$ with $l_1, l_2 \geq n$. If for instance $l_2 \geq l_1$, by the hypothesis on \mathfrak{b} , we can write $b = \sum_{i=1}^k b'_i b''_i$ for $b'_i \in \mathfrak{b}(l_1, m)$ and $b''_i \in \mathfrak{b}(l_2, l_1)$ and hence $a \otimes b = \sum_{i=1}^k (a \otimes b'_i)(1 \otimes b''_i) \in \Delta S$. \square

4.3. Quasicoherent sheaves on projective schemes. Next we apply the results of §4.2 to graded algebras and schemes. A *graded algebra* $A = (A_n)_{n \in \mathbb{N}}$ is an algebra $A = \bigoplus_{n \in \mathbb{N}} A_n$ with $1 \in A_0$ and multiplication determined by $A_n \otimes A_m \rightarrow A_{n+m}$. Such an algebra has an associated \mathbb{Z} -algebra $\mathfrak{a}(A)$ with $\mathfrak{a}(A)(n, m) = A_{n-m}$. The algebra A is generated in degrees $D \subseteq \mathbb{N}$ (resp. finitely generated, resp. connected) if and only if the associated \mathbb{Z} -algebra $\mathfrak{a}(A)$ is. Now if A is a finitely generated, connected graded algebra, the category $\text{Gr}(A)$ of graded A -modules has a localizing subcategory $\text{Tors}(A)$ of torsion modules, and one obtains the quotient category $\text{Qgr}(A) = \text{Gr}(A)/\text{Tors}(A)$. By Serre's theorem, if A is commutative with associated projective scheme $\text{Proj}(A)$, we have $\text{Qch}(\text{Proj}(A)) = \text{Qgr}(A)$. The category $\text{Qgr}(A)$ has been generalized to certain classes of \mathbb{Z} -algebras in [38], [40], [32] and in [11], the category $\text{Sh}(\mathfrak{a}, \mathcal{T}_{\text{tails}})$ is introduced as a further generalization to arbitrary \mathbb{Z} -algebras. In particular, for a finitely generated connected graded algebra A , we have

$$(15) \quad \text{Qgr}(A) \cong \text{Sh}(\mathfrak{a}(A), \mathcal{T}_{\text{tails}})$$

Next we turn to tensor products. For two graded algebras A and B , the *cartesian product* $A \times_{\text{cart}} B$ is defined by $(A \times_{\text{cart}} B)_n = A_n \otimes B_n$. We clearly have

$$(16) \quad \mathfrak{a}(A \times_{\text{cart}} B) = (\mathfrak{a}(A) \otimes \mathfrak{a}(B))_{\Delta}.$$

Theorem 4.12. (1) *For two graded algebras A and B which are connected and finitely generated in degree 1, we have*

$$\text{Qgr}(A) \boxtimes \text{Qgr}(B) = \text{Qgr}(A \times_{\text{cart}} B).$$

(2) *For two projective schemes X and Y , we have*

$$\text{Qch}(X) \boxtimes \text{Qch}(Y) = \text{Qch}(X \times Y).$$

Proof. (1) Put $\mathfrak{a} = \mathfrak{a}(A)$, $\mathfrak{b} = \mathfrak{a}(B)$. According to (15) and Theorem 4.9, we have

$$\text{Qgr}(A) \boxtimes \text{Qgr}(B) \cong \text{Sh}(\mathfrak{a}, \mathcal{T}_{\text{tails}}) \boxtimes \text{Sh}(\mathfrak{b}, \mathcal{T}_{\text{tails}}) \cong \text{Sh}((\mathfrak{a} \otimes \mathfrak{b})_{\Delta}, \mathcal{T}_{\text{tails}})$$

and by (16) and (15), the category on the right hand side is isomorphic to $\text{Qgr}(A \times_{\text{cart}} B)$. (2) It suffices to write $X \cong \text{Proj}(A)$ and $Y \cong \text{Proj}(B)$ for connected graded algebras generated in degree 1. \square

Remark 4.13. The formula $\mathrm{Qch}(X) \boxtimes \mathrm{Qch}(Y) = \mathrm{Qch}(X \times Y)$ should hold in greater generality, at least for schemes and suitable stacks. This will follow from the appropriate compatibility between tensor products and descent, and is work in progress.

5. RELATION WITH OTHER TENSOR PRODUCTS

Our tensor product of Grothendieck categories is in close relation with two well-known tensor products of categories. In this section we analyse those relations.

The first one is the tensor product of locally presentable categories. It is well-known that every Grothendieck category is locally presentable. In §5.1 we prove that taking our tensor product of two Grothendieck categories coincides with taking their tensor product as locally presentable categories. In particular, the class of locally α -presentable Grothendieck categories for a fixed cardinal α is stable under our tensor product. This applies, for example, to the class of locally finitely presentable Grothendieck categories. This should be contrasted with the more restrictive class of locally coherent Grothendieck categories, which is not preserved, as is already clear from the ring case.

The second one is Deligne's tensor product of small abelian categories. In §5.2, for small abelian categories \mathcal{A} and \mathcal{B} with associated Grothendieck categories $\mathrm{Lex}(\mathcal{A})$ and $\mathrm{Lex}(\mathcal{B})$ of left exact modules, based upon [24] the tensor product $\mathrm{Lex}(\mathcal{A}) \boxtimes \mathrm{Lex}(\mathcal{B})$ is shown to be locally coherent precisely when the Deligne tensor product of \mathcal{A} and \mathcal{B} exists, and in this case the Deligne tensor product is given by the abelian category of finitely presented objects in $\mathrm{Lex}(\mathcal{A}) \boxtimes \mathrm{Lex}(\mathcal{B})$. Following a suggestion by Henning Krause, in §5.3 we define an α -version of the Deligne tensor product which is shown to underlie any given tensor product of Grothendieck categories, as long as we choose α sufficiently large.

5.1. Tensor product of locally presentable categories. Local presentability of categories is classically considered in a non-enriched context [15], for which enriched analogues exist [21]. In the case of k -linear categories, where enrichment is over $\mathrm{Mod}(k)$, the classical and the enriched notions of local presentability coincide. For the constructions considered in this section though, it is essential to work enriched over $\mathrm{Mod}(k)$. All categories and constructions in this section are to be understood in the k -linear sense.

Recall that a k -linear category \mathcal{C} is *locally presentable* if it is cocomplete and there exists a small regular cardinal α such that \mathcal{C} has a set of strong generators consisting of *α -presented objects*, that is objects $G \in \mathcal{C}$ such that the k -linear functor $\mathcal{C}(G, -) : \mathcal{C} \rightarrow \mathrm{Mod}(k)$ preserves α -filtered colimits. In this case the full subcategory \mathcal{C}_α of α -presented objects is small, α -cocomplete (i.e. closed under α -small colimits) and it is obtained as the closure of the category of generators under α -small colimits [21]. When we want to make the cardinal α explicit we will say \mathcal{C} is *locally α -presentable*. Observe that this notion is a generalization to bigger cardinals of the notion of *locally finitely presentable k -linear category*, which is obtained as the particular case with $\alpha = \aleph_0$. In that case we write $\mathrm{fp}(\mathcal{C}) = \mathcal{C}_{\aleph_0}$.

It is well known that Grothendieck categories are locally presentable (see for example [6, Prop 3.4.16]).

Consider k -linear categories \mathcal{A} , \mathcal{B} and \mathcal{C} . We denote by $\mathrm{Cont}(\mathcal{A}, \mathcal{B})$ (resp. by $\mathrm{Cont}_\alpha(\mathcal{A}, \mathcal{B})$) the category of k -linear continuous (resp. α -continuous) functors from \mathcal{A} to \mathcal{B} , that is functors preserving all (existing) limits (resp. α -small limits). We denote by $\mathrm{Cont}(\mathcal{A}, \mathcal{B}; \mathcal{C})$ (resp. by $\mathrm{Cont}_\alpha(\mathcal{A}, \mathcal{B}; \mathcal{C})$) the category of functors $\mathcal{A} \times \mathcal{B} \rightarrow \mathcal{C}$ which are k -linear and continuous in each variable.

The categories $\mathrm{Cocont}(\mathcal{A}, \mathcal{B})$, $\mathrm{Cocont}_\alpha(\mathcal{A}, \mathcal{B})$, $\mathrm{Cocont}(\mathcal{A}, \mathcal{B}; \mathcal{C})$ and $\mathrm{Cocont}_\alpha(\mathcal{A}, \mathcal{B}; \mathcal{C})$ are defined similarly with limits replaced by colimits.

In the following theorem a tensor product of locally presentable categories is described.

Theorem 5.1. [7, Lem. 2.6, Rem. 2.7], [9, §2], [10, Cor. 2.2.5] *Consider locally presentable k -linear categories \mathcal{A} and \mathcal{B} .*

- (1) *The category $\text{Cocont}(\mathcal{A}, \mathcal{B})$ of k -linear cocontinuous functors is also a locally presentable k -linear category.*
- (2) *There exists a locally presentable k -linear category $\mathcal{A} \boxtimes_{\text{LP}} \mathcal{B}$ such that for every cocomplete k -linear category \mathcal{C} there is a natural equivalence of categories:*

$$\text{Cocont}(\mathcal{A} \boxtimes_{\text{LP}} \mathcal{B}, \mathcal{C}) \cong \text{Cocont}(\mathcal{A}, \mathcal{B}; \mathcal{C}) \cong \text{Cocont}(\mathcal{A}, \text{Cocont}(\mathcal{B}, \mathcal{C}))$$

- (3) *In (2) we can take $\mathcal{A} \boxtimes_{\text{LP}} \mathcal{B} = \text{Cont}(\mathcal{A}^{\text{op}}, \mathcal{B})$.*

For small α -cocomplete k -linear categories \mathfrak{c} and \mathfrak{d} , we put

$$\text{Lex}_\alpha(\mathfrak{c}) = \text{Cont}_\alpha(\mathfrak{c}^{\text{op}}, \text{Mod}(k)) \subseteq \text{Mod}(\mathfrak{c})$$

and

$$\text{Lex}_\alpha(\mathfrak{c}, \mathfrak{d}) = \text{Cont}_\alpha(\mathfrak{c}^{\text{op}}, \mathfrak{d}^{\text{op}}; \text{Mod}(k)) \subseteq \text{Mod}(\mathfrak{c} \otimes \mathfrak{d}).$$

For $\alpha = \aleph_0$, we obtain the familiar categories $\text{Lex}(\mathfrak{c}) = \text{Lex}_{\aleph_0}(\mathfrak{c})$ of left exact (that is, finite limit preserving) modules and $\text{Lex}(\mathfrak{c}, \mathfrak{d}) = \text{Lex}_{\aleph_0}(\mathfrak{c}, \mathfrak{d})$ of modules that are left exact in both variables.

The category $\text{Lex}_\alpha(\mathfrak{c})$ is locally α -presentable, and we have $(\text{Lex}_\alpha(\mathfrak{c}))_\alpha \cong \mathfrak{c}$. The category $\text{Lex}_\alpha(\mathfrak{c})$ is the α -free cocompletion of \mathfrak{c} : every object in it can be written as an α -filtered colimit of \mathfrak{c} -objects, and according to [21, Thm. 9.9], for any cocomplete k -linear category \mathcal{D} we have

$$(17) \quad \text{Cocont}(\text{Lex}_\alpha(\mathfrak{c}), \mathcal{D}) = \text{Cocont}_\alpha(\mathfrak{c}, \mathcal{D}).$$

Conversely, for a locally α -presentable k -linear category \mathcal{C} , according to [21, Thm. 7.2 + §7.4], we have

$$(18) \quad \mathcal{C} \cong \text{Lex}_\alpha(\mathcal{C}_\alpha).$$

One thus also obtains a natural α -cocomplete tensor product for small α -cocomplete k -linear categories \mathfrak{c} and \mathfrak{d} [21], [22], given by

$$\mathfrak{c} \otimes_\alpha \mathfrak{d} = (\text{Lex}_\alpha(\mathfrak{c}) \boxtimes_{\text{LP}} \text{Lex}_\alpha(\mathfrak{d}))_\alpha.$$

This α -cocomplete tensor product satisfies the following universal property for every small α -cocomplete k -linear category \mathfrak{e} :

$$(19) \quad \text{Cocont}_\alpha(\mathfrak{c} \otimes_\alpha \mathfrak{d}, \mathfrak{e}) \cong \text{Cocont}_\alpha(\mathfrak{c}, \mathfrak{d}; \mathfrak{e}).$$

For small finitely cocomplete categories \mathfrak{c} and \mathfrak{d} , we denote $\mathfrak{c} \otimes_{\text{fp}} \mathfrak{d} = \mathfrak{c} \otimes_{\aleph_0} \mathfrak{d}$.

The following alternative description of the tensor product of locally presentable categories is useful for our purpose. It appears for example in [9]; we provide a proof for the convenience of the reader.

Proposition 5.2. *For locally α -presentable k -linear categories \mathcal{C} and \mathcal{D} , we have an equivalence*

$$\mathcal{C} \boxtimes_{\text{LP}} \mathcal{D} \cong \text{Lex}_\alpha(\mathcal{C}_\alpha, \mathcal{D}_\alpha).$$

Proof. We have equivalences

$$\begin{aligned}
 \mathcal{C} \boxtimes_{\text{LP}} \mathcal{D} &= \text{Cont}(\mathcal{C}^{\text{op}}, \mathcal{D}) \\
 &\cong \text{Cocont}(\mathcal{C}, \mathcal{D}^{\text{op}})^{\text{op}} \\
 &\cong \text{Cocont}(\text{Lex}_{\alpha}(\mathcal{C}_{\alpha}), \mathcal{D}^{\text{op}})^{\text{op}} \\
 &\cong \text{Cocont}_{\alpha}(\mathcal{C}_{\alpha}, \mathcal{D}^{\text{op}})^{\text{op}} \\
 &\cong \text{Cont}_{\alpha}(\mathcal{C}_{\alpha}^{\text{op}}, \text{Cont}_{\alpha}(\mathcal{D}_{\alpha}^{\text{op}}, \text{Mod}(k))) \\
 &\cong \text{Lex}_{\alpha}(\mathcal{C}_{\alpha}, \mathcal{D}_{\alpha}),
 \end{aligned}$$

where we have used (18) in the third and fifth steps, (17) in the fourth step, and the fact that limits are computed pointwise in $\text{Cont}_{\alpha}(\mathcal{D}_{\alpha}^{\text{op}}, \text{Mod}(k))$ in the last step. \square

Next we turn our attention to (k -linear) Grothendieck categories. The following result combines [26, Thm 3.7] and [21, Thm 7.2 + §7.4], the latter being the enriched version of the analogous classical statement already formulated in [15]:

Theorem 5.3. *Let \mathcal{C} be a locally α -presentable Grothendieck category and consider the inclusion $u_{\mathcal{C}} : \mathcal{C}_{\alpha} \rightarrow \mathcal{C}$. The canonical functor $\mathcal{C} \rightarrow \text{Mod}(\mathcal{C}_{\alpha}) : C \mapsto \mathcal{C}(u(-), C)$ factors through an equivalence of categories*

$$\mathcal{C} \rightarrow \text{Sh}(\mathcal{C}_{\alpha}, u_{\mathcal{C}}^{-1}\mathcal{T}_{\mathcal{C}}) = \text{Lex}_{\alpha}(\mathcal{C}_{\alpha}).$$

We can now prove the main result of this section:

Theorem 5.4. *For Grothendieck categories \mathcal{C} and \mathcal{D} , we have an equivalence of categories*

$$\mathcal{C} \boxtimes \mathcal{D} \cong \mathcal{C} \boxtimes_{\text{LP}} \mathcal{D}.$$

Proof. Let α be a regular cardinal for which both \mathcal{C} and \mathcal{D} are locally α -presentable. By Theorem 5.3, we have

$$\mathcal{C} \boxtimes \mathcal{D} = \text{Sh}(\mathcal{C}_{\alpha} \otimes \mathcal{D}_{\alpha}, u_{\mathcal{C}}^{-1}\mathcal{T}_{\mathcal{C}} \boxtimes u_{\mathcal{D}}^{-1}\mathcal{T}_{\mathcal{D}}) = \text{Lex}_{\alpha}(\mathcal{C}_{\alpha}, \mathcal{D}_{\alpha})$$

since $\text{Lex}_{\alpha}(\mathcal{C}_{\alpha}, \mathcal{D}_{\alpha})$ describes the intersection of the two one-sided sheaf categories following Theorem 5.3. This finishes the proof by Proposition 5.2. \square

Remark 5.5. The way in which the tensor product \boxtimes_{LP} of locally presentable categories is defined through a universal property, makes it well-defined up to equivalence of categories. As an alternative to our independent approach to the tensor product of Grothendieck categories based upon functoriality, one can show in the spirit of Proposition 5.2 that $\text{Sh}(\mathfrak{a}, \mathcal{T}_{\mathfrak{a}}) \boxtimes_{\text{LP}} \text{Sh}(\mathfrak{b}, \mathcal{T}_{\mathfrak{b}}) = \text{Sh}(\mathfrak{a} \otimes \mathfrak{b}, \mathcal{T}_{\mathfrak{a}} \boxtimes \mathcal{T}_{\mathfrak{b}})$.

Corollary 5.6. *The subclass of Grothendieck k -linear categories within the class of locally presentable k -linear categories is closed under the tensor product \boxtimes_{LP} .*

5.2. Relation with Deligne's tensor product. In [12], Deligne defined a tensor product for abelian categories through a universal property. This tensor product is known to exist only under additional assumptions on the categories. Recall that a Grothendieck category \mathcal{C} is *locally coherent* if it is locally finitely presentable and $\text{fp}(\mathcal{C})$ is abelian. This defines a 1-1 correspondence between locally coherent Grothendieck categories on the one hand and small abelian categories on the other hand, the inverse being given by $\mathcal{A} \mapsto \text{Lex}(\mathcal{A})$. For small abelian categories \mathcal{A} and \mathcal{B} , according to §5.1 we have

$$\text{Lex}(\mathcal{A}) \boxtimes \text{Lex}(\mathcal{B}) = \text{Lex}(\mathcal{A}, \mathcal{B}).$$

Since the tensor product of coherent rings is not necessarily coherent (see for instance [24, Ex. 21]), the tensor product of locally coherent Grothendieck categories is not necessarily locally coherent. We can complete [24, Thm. 18] as follows:

Theorem 5.7. *For small abelian categories \mathcal{A} and \mathcal{B} , the following are equivalent:*

- (1) *Deligne's tensor product of \mathcal{A} and \mathcal{B} exists;*
- (2) *The tensor product $\mathcal{A} \otimes_{\text{fp}} \mathcal{B}$ is abelian;*
- (3) *The tensor product $\text{Lex}(\mathcal{A}) \boxtimes \text{Lex}(\mathcal{B})$ is locally coherent.*

In this case, Deligne's tensor product equals $\mathcal{A} \otimes_{\text{fp}} \mathcal{B} = \text{fp}(\text{Lex}(\mathcal{A}) \boxtimes \text{Lex}(\mathcal{B}))$.

5.3. The α -Deligne tensor product. As suggested to us by Henning Krause, we define an α -version of the Deligne tensor product for a cardinal α and we show that every tensor product of Grothendieck categories is accompanied by a parallel α -Deligne tensor product of its categories of α -presented objects for sufficiently large α .

Definition 5.8. (1) Let \mathcal{A} and \mathcal{B} be α -cocomplete abelian categories. An α -Deligne tensor product of \mathcal{A} and \mathcal{B} is an α -cocomplete abelian category $\mathcal{A} \bullet_{\alpha} \mathcal{B}$ with a functor $\mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A} \bullet_{\alpha} \mathcal{B}$ which is α -cocontinuous in each variable and induces equivalences

$$\text{Cocont}_{\alpha}(\mathcal{A} \bullet_{\alpha} \mathcal{B}, \mathcal{C}) \cong \text{Cocont}_{\alpha}(\mathcal{A}, \mathcal{B}; \mathcal{C})$$

for every α -cocomplete abelian category \mathcal{C} .

- (2) Let \mathcal{A} and \mathcal{B} be abelian categories. If it exists, we define the *modified α -Deligne tensor product* to be

$$\mathcal{A} \tilde{\bullet}_{\alpha} \mathcal{B} = \text{Lex}(\mathcal{A})_{\alpha} \bullet_{\alpha} \text{Lex}(\mathcal{B})_{\alpha}.$$

Note that for $\alpha = \aleph_0$, we have $\mathcal{A} \tilde{\bullet}_{\aleph_0} \mathcal{B} = \mathcal{A} \bullet_{\aleph_0} \mathcal{B} = \mathcal{A} \bullet \mathcal{B}$.

The following is proven along the lines of [19, Prop. 6.1.13], using the description of $\text{Lex}_{\alpha}(\mathcal{A}) \cong \text{Ind}_{\alpha}(\mathcal{A})$ as “ind completion” in terms of α -filtered colimits.

Lemma 5.9. *Let \mathcal{A} be a small α -cocomplete abelian category. The category $\text{Lex}_{\alpha}(\mathcal{A})$ is abelian.*

The following analogue of [24, Lem. 17] is proven along the same lines, based upon Lemma 5.9.

Lemma 5.10. *Suppose for small α -cocomplete abelian categories \mathcal{A} and \mathcal{B} , the α -Deligne tensor product $\mathcal{A} \bullet_{\alpha} \mathcal{B}$ exists. The category $\text{Lex}_{\alpha}(\mathcal{A} \bullet_{\alpha} \mathcal{B})$ is characterized by the following universal property for cocomplete abelian categories \mathcal{C} :*

$$(20) \quad \text{Cocont}(\text{Lex}_{\alpha}(\mathcal{A} \bullet_{\alpha} \mathcal{B}), \mathcal{C}) = \text{Cocont}_{\alpha}(\mathcal{A}, \mathcal{B}; \mathcal{C}).$$

The following replacement of [24, Thm. 18] is proven along the same lines. For $\alpha = \aleph_0$, note that the second part of condition (1) is automatically fulfilled.

Theorem 5.11. *For α -cocomplete abelian categories \mathcal{A} and \mathcal{B} , the following are equivalent:*

- (1) *The α -Deligne tensor product $\mathcal{A} \bullet_{\alpha} \mathcal{B}$ exists and $\text{Lex}_{\alpha}(\mathcal{A} \otimes_{\alpha} \mathcal{B})$ is abelian;*
- (2) *The α -cocomplete tensor product $\mathcal{A} \otimes_{\alpha} \mathcal{B}$ is abelian.*

In this case, we have $\mathcal{A} \bullet_{\alpha} \mathcal{B} = \mathcal{A} \otimes_{\alpha} \mathcal{B}$.

Proof. If $\mathcal{A} \otimes_{\alpha} \mathcal{B}$ is abelian, it obviously satisfies the universal property of $\mathcal{A} \bullet_{\alpha} \mathcal{B}$ and further, $\text{Lex}_{\alpha}(\mathcal{A} \otimes_{\alpha} \mathcal{B})$ is abelian by Lemma 5.9. Conversely, suppose $\mathcal{A} \bullet_{\alpha} \mathcal{B}$ exists and $\text{Lex}_{\alpha}(\mathcal{A} \otimes_{\alpha} \mathcal{B})$ is abelian. The categories $\text{Lex}_{\alpha}(\mathcal{A} \bullet_{\alpha} \mathcal{B})$ and $\text{Lex}_{\alpha}(\mathcal{A} \otimes_{\alpha} \mathcal{B})$ have the categories $\mathcal{A} \bullet_{\alpha} \mathcal{B}$ and $\mathcal{A} \otimes_{\alpha} \mathcal{B}$ as respective categories of α -presented objects, whence it suffices to show that $\text{Lex}_{\alpha}(\mathcal{A} \otimes_{\alpha} \mathcal{B})$, being cocomplete and abelian by assumption, has the universal property of Lemma 5.10. But this is clearly the case by (17) (19). \square

We recall the following:

Proposition 5.12. [23, Cor. 5.2] *Let \mathcal{C} be a Grothendieck category. There exists a cardinal α such that \mathcal{C} is locally β -presentable and \mathcal{C}_β is abelian for $\beta \geq \alpha$.*

Whereas the tensor product of two Grothendieck categories cannot be related to the Deligne tensor product in general, it can always be related to an α -Deligne tensor product in the following way:

Proposition 5.13. *Let \mathcal{C} and \mathcal{D} be Grothendieck categories. There exists a cardinal α such that for $\beta \geq \alpha$ the β -Deligne tensor product $\mathcal{C}_\beta \bullet_\beta \mathcal{D}_\beta$ exists and we have*

$$(\mathcal{C} \boxtimes \mathcal{D})_\beta \cong \mathcal{C}_\beta \bullet_\beta \mathcal{D}_\beta.$$

Proof. It suffices to note that by Proposition 5.12, we can choose α such that for $\beta \geq \alpha$ the categories \mathcal{C} , \mathcal{D} and $\mathcal{C} \boxtimes \mathcal{D}$ are locally β -presentable and \mathcal{C}_β , \mathcal{D}_β and $(\mathcal{C} \boxtimes \mathcal{D})_\beta$ are abelian. Hence, we have $\mathcal{C}_\beta \otimes_\beta \mathcal{D}_\beta \cong (\mathcal{C} \boxtimes \mathcal{D})_\beta$ by Theorem 5.4, and thus the desired isomorphism holds by Theorem 5.11. \square

As a special case, whereas two small abelian categories do not necessarily have a Deligne tensor product, they do have a modified α -Deligne tensor product for sufficiently large α :

Corollary 5.14. *Let \mathcal{A} and \mathcal{B} be small abelian categories. There exists a cardinal α such that for $\beta \geq \alpha$ the modified β -Deligne tensor product $\mathcal{A} \bullet_\beta \mathcal{B}$ exists and we have*

$$(\text{Lex}(\mathcal{A}, \mathcal{B}))_\beta \cong \mathcal{A} \bullet_\beta \mathcal{B}.$$

5.4. Relation with the tensor product of toposes and future prospects.

In Theorem 5.4 we have shown that the tensor product of Grothendieck categories is a special instance of the tensor product of locally presentable linear categories, using special linear site presentations of the categories. This raises the natural question whether, if one takes the tensor product of locally presentable categories as starting point, there is a shorter route to the tensor product of Grothendieck categories than the one we followed.

First one may note that in order to obtain an abstract tensor product of Grothendieck categories, it suffices to prove Corollary 5.6 directly. As an anonymous referee suggested, one can prove along the lines of [24, Cor. 15] that the tensor product of locally presentable categories preserves the Grothendieck property. However, this does not bring us any closer to the concrete expressions of the tensor product in terms of arbitrary representations in terms of linear sites, which is the main aim of the current paper.

Secondly one may note that, after proposing our concrete formula for the tensor product of Grothendieck categories using linear sites, it suffices to show that this formula satisfies the universal property of the tensor product of locally presentable linear categories in order to show at once that our formula leads to a good definition, and that Theorem 5.4 holds. This approach indeed works, and is based upon the possibility to write down an analogous formula to (17), with regard to a linear site $(\mathfrak{a}, \mathcal{T})$. Precisely, for any cocomplete k -linear category \mathcal{D} we have

$$(21) \quad \text{Cocont}(\text{Sh}(\mathfrak{a}, \mathcal{T}), \mathcal{D}) = \text{Cocont}_{\mathcal{T}}(\mathfrak{a}, \mathcal{D})$$

where the right hand side denotes the category of linear functors $F : \mathfrak{a} \rightarrow \mathcal{D}$ whose induced colimit preserving functor $\hat{F} : \text{Mod}(\mathfrak{a}) \rightarrow \mathcal{D}$ sends inclusions of covers $R \subseteq \mathfrak{a}(-, A)$ to isomorphisms in \mathcal{D} . Rather than spelling out the proof of the universal property for the tensor product in Definition 4.2, we refer the reader to [31] where the parallel reasoning is performed for toposes over Set . The tensor product of Grothendieck categories which we have introduced can be seen as a linear counterpart to the *product* of Grothendieck toposes which is described by Johnstone in [16], [17]. In [31], Pitts shows that the product of Grothendieck

toposes is a special instance of the (**Set**-based) tensor product of locally presentable categories, using the universal property.

Unlike in the case of toposes, working over $\text{Mod}(k)$ rather than over **Set**, our tensor product does not describe a 2-categorical product, but instead introduces a 2-categorical monoidal structure on linear toposes. The natural way of defining a product (Grothendieck) topology as the smallest topology making the projections continuous is not available in the linear setup, as there are no projection morphisms to the factors of the tensor product. Hence our definition of the product site in §2.4 is necessarily somewhat ad hoc. Further, we should note that the establishment of the correct formula for the tensor product does not automatically yield the tangle functoriality properties for linear sites which we have proven. With our motivation coming from non-commutative geometry, it is precisely the flexibility in choosing appropriate sites, and the possibility to view certain functors of geometric origin as induced by natural morphisms of sites, which is of greatest interest to us.

The notion of LC morphism which we prove in Proposition 3.14 to be stable under the tensor product, is more restrictive than a morphism inducing an equivalence on the level of sheaf categories, and so this result cannot be deduced a posteriori from the existence of the tensor product satisfying the universal property. In fact, the class of LC morphisms opens up the interesting possibility to describe the “category of Grothendieck categories” up to equivalence as a 2-category of fractions, obtained from the category of linear sites by inverting LC morphisms. This fact, and its implications for the tensor product, will be elaborated further in [35].

On the other hand, a combination of Pitts’ approach and our description of the tensor product in terms of localizing Serre subcategories leads to a natural tensor product for well-generated algebraic triangulated categories, which stand in relation to derived categories of differential graded algebras like Grothendieck categories stand in relation to module categories according to [34]. To make this idea precise, one takes Toën’s inner hom between dg categories in the homotopy category as starting point, and between (homologically) cocomplete (with respect to arbitrary set indexed coproducts) dg categories one considers its restriction $\text{RHom}_c(\mathcal{A}, \mathcal{B})$ to bimodules inducing cocontinuous functors on the level of homology, inspired upon [39, §7]. The *cocomplete pretriangulated tensor product* between cocomplete pretriangulated dg categories \mathcal{A} and \mathcal{B} is by definition, if it exists, the unique cocomplete pretriangulated dg category $\mathcal{A} \boxtimes_c \mathcal{B}$ satisfying the following universal property with respect to cocomplete pretriangulated dg categories \mathcal{C} :

$$(22) \quad \text{RHom}_c(\mathcal{A} \boxtimes_c \mathcal{B}, \mathcal{C}) \cong \text{RHom}_c(\mathcal{A}, \text{RHom}_c(\mathcal{B}, \mathcal{C})).$$

In the dg world, dg topologies are not quite the right tool in order to perform localization on the derived level. We define a *dg site* as a small dg category \mathfrak{a} along with a localizing thick subcategory $\mathcal{W} \subseteq D(\mathfrak{a})$ of the derived category. If by $\underline{D}(\mathfrak{a})$ we denote the dg derived category, then the dg quotient $\underline{D}(\mathfrak{a})$ by \mathcal{W} can be characterized by the following replacement of (21)

$$(23) \quad \text{RHom}_c(\underline{D}(\mathfrak{a})/\mathcal{W}, \mathcal{C}) \cong \text{RHom}_{\mathcal{W}}(\mathfrak{a}, \mathcal{C})$$

where the right hand side denotes the subcategory of $\text{RHom}(\mathfrak{a}, \mathcal{C})$ consisting of the bimodules for which the induced cocontinuous functor $D(\mathfrak{a}) \rightarrow H^0(\mathcal{C})$ sends \mathcal{W} to zero. With a definition inspired upon §2.5, one can define the tensor product of well-generated triangulated categories and show that it satisfies the universal property (22). The development of this approach, as well as its precise relation to the tensor product of Grothendieck categories, in particular under suitable flatness hypothesis like the one from [28], are work in progress and will appear in [35]. Further, the definition should also be related to the tensor product of locally presentable infinity categories [29, §4.1].

In the future, it is our goal to apply tensor products of large categories (Grothendieck or enhanced triangulated) in order to widen the scope of the generalized Deligne conjecture proven in [36], [37].

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